WEAK TYPE ESTIMATES FOR FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES

I. Sihwaningrum and Y. Sawano∗

Abstract
We discuss here a weak type estimate for fractional integral operators on Morrey spaces, where the underlying measure $\mu$ does not always satisfy the doubling condition.

Keywords: Weak type estimates, fractional integral operators, Morrey spaces, non-doubling measure

2000 Mathematics Subject Classification: 42B20, 26A33, 47B38, 47G10.

1 Introduction
The aim of this paper is to propose a framework of Morrey spaces and fractional integral operators when we are given a Radon measure $\mu$ on a metric measure space $(X, d, \mu)$.

On $\mathbb{R}^d$, recall that the Riesz potential $I_\alpha$ is given by

$$ I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-\alpha}} \, dy. $$

According to the Hardy-Littlewood-Sobolev theorem [1, 2, 8], $I_\alpha$ is known to be bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ as long as $p, q \in (1, \infty)$ satisfies $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Morrey spaces, named after C. Morrey, seem to describe the boundedness property of $I_\alpha$. Here we adopt the following notation to denote Morrey spaces. Let $1 \le q \le p < \infty$. For a measurable function $f$ on $\mathbb{R}^d$, we define

$$ \|f\|_{M^p_q} := \sup \left\{ |B|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(B)} : B \text{ is a ball} \right\}. $$

The space $\mathcal{M}^p_q(\mathbb{R}^d)$ denotes the set of all measurable functions $f$ for which the norm $\|f\|_{\mathcal{M}^p_q}$ is finite. According to Adams, $I_\alpha$ is bounded from $\mathcal{M}^p_q(\mathbb{R}^d)$ to $\mathcal{M}^s_t(\mathbb{R}^d)$, provided $p, q, s, t \in (1, \infty)$ satisfies $\frac{p}{q} = \frac{s}{t}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Although the integral kernel does not belong to any $L^r(\mathbb{R}^d)$-spaces with $1 \le r \le \infty$, we still have such a boundedness.

In this paper, we aim to show that this theorem is independent from the geometric structure of $\mathbb{R}^d$ by extending it to metric measure spaces, where all we have is the distance function $d$ and the Radon measure $\mu$. 

1
Let \((X, d, \mu)\) be a metric measure space with a distance function \(d\) and a Borel measure \(\mu\). Recall that the measure \(\mu\) is a doubling measure if it satisfies the so-called doubling condition, that is, there exists a constant \(C > 0\) such that
\[
\mu(B(a, 2r)) \leq C\mu(B(a, r))
\]
for every ball \(B(a, r)\) with center \(a \in X\) and radius \(r > 0\). The doubling condition had been a key property in classical harmonic analysis but around a decade ago, it turned out to be unnecessary. The point is that we modify the related definitions. Indeed, we propose, in the present paper, it suffices to redefine the fractional integral operator by
\[
I_\alpha f(x) := \int_X \frac{f(y)}{\mu(B(x, 2d(x, y)))^{1-\alpha}} \, d\mu(y).
\]
Note that the definition is made independent of any notion of dimensions. The same can be said for Morrey spaces, which we shall define now. For \(k > 0, 1 \leq p < \infty\) and \(f \in L^1_{\text{loc}}(\mu)\), the norm is given by
\[
\|f\|_{\mathcal{M}_{p}^{s}(k, \mu)} := \sup \left\{ \mu(B(x, kr))^{1/p-1}\|\chi_{B(x,r)}f\|_{L^1(\mu)} : x \in X, r > 0, \mu(B(x, r)) > 0 \right\},
\]
where \(\chi_{B(x,r)}\) denotes the characteristic function of the ball \(B(x, r)\).

We will prove here that \(I_\alpha\) satisfies a weak type estimate on Morrey spaces \(\mathcal{M}_1^p(\mu)\). Our main results are:

**Theorem 1.1.** If \(1 < p < \infty, 1 < s < \infty, 0 < \alpha < \frac{1}{p}\) and \(\frac{1}{s} = \frac{1}{p} - \alpha\), then there exist \(C > 0\) such that
\[
\mu\{x \in B(a, r) : I_\alpha f(x) > \gamma\} \leq C\mu(B(a, r))^{1-1/p}\left(\frac{\|f\|_{\mathcal{M}_{p}^{s}(\mu)}}{\gamma}\right)^{s/p}
\]
for all positive \(\mu\)-measurable functions \(f\).

and

**Theorem 1.2.** If \(1 < q \leq p < \infty, 1 < s < \infty, 0 < \alpha < \frac{1}{p}, \frac{q}{p} = \frac{t}{s}\) and \(\frac{1}{s} = \frac{1}{p} - \alpha\), then there exist \(C > 0\) such that
\[
\|I_\alpha f\|_{\mathcal{M}_{p}^{t}(\mu)} \leq C\|f\|_{\mathcal{M}_{p}^{s}(\mu)}
\]
for all positive \(\mu\)-measurable functions \(f\).
2 Main Results

We define the centered maximal operator
\[ M_k f(x) := \sup_{r > 0} \frac{1}{\mu(B(x, kr))} \int_{B(x, r)} |f(y)| \, d\mu(y) \quad (x \in \text{supp}(\mu)). \]

About the maximal operator \( M_2 \), we prove the following boundedness on Morrey spaces.

**Theorem 2.1.** For any \( \gamma > 0 \), positive \( \mu \)-measurable functions and any ball \( B(a, r) \), inequality
\[ \mu\{x \in B(a, r) : M_2 f(x) > \gamma \} \leq 4 \frac{\mu(B(a, 6r))^{1 - 1/p}}{\gamma} \|f\|_{M_1^p(2, \mu)} \]
holds.

**Proof.** We actually prove
\[ \mu\{x \in B(a, r) : M_2 f(x) > 2\gamma \} \leq 2 \frac{\mu(B(a, 6r))^{1 - 1/p}}{\gamma} \|f\|_{M_1^p(2, \mu)}. \]

Once we prove
\[ \mu\{x \in B(a, r) : M_2 [\chi_{B(a, 3r)} f](x) > \gamma \} \leq \frac{\mu(B(a, 6r))^{1 - 1/p}}{\gamma} \|f\|_{M_1^p(2, \mu)}, \]
and
\[ \mu\{x \in B(a, r) : M_2 [\chi_{B(a, r)} f](x) > \gamma \} \leq \frac{\mu(B(a, 6r))^{1 - 1/p}}{\gamma} \|f\|_{M_1^p(2, \mu)}, \]
then estimate (3) follows automatically. Estimate (4) follows from the weak-\( L^1(\mu) \) boundedness of \( M_2 \) (see [7, 9]).

Denote by \( B(\mu) \) the set of all balls with positive \( \mu \)-measure. A geometric observation shows that
\[ M_2 [\chi_{B(a, 3r)} f](x) \leq \sup_{B \in B(\mu), B \cap B(a, r) \neq \emptyset} \frac{1}{\mu(2B)} \int_B |f(y)| \, d\mu(y). \]

Let \( B \) be a ball which intersects both \( B(a, r) \) and \( X \setminus B(a, 3r) \). The ball \( B \) engulfs \( B(a, r) \) if we double the radius of \( B \). Thus,
\[ \mu(B(a, 6r))^{1/p - 1} \mu\{x \in B(a, r) : M_2 [\chi_{B(a, 3r)} f](x) > \gamma \} \]
\[ \leq \mu(B(a, r))^{1/p - 1} \sup_{B \in B(\mu), B \cap B(a, r) \neq \emptyset} \frac{1}{\mu(2B)} \int_B |f(y)| \, d\mu(y) \]
\[ \leq \sup_{B \in B(\mu), B \cap B(a, r) \neq \emptyset, B \cap (X \setminus B(a, 3r)) \neq \emptyset, \mu(2B) \neq 0} \]
\[ \frac{\mu(2B)^{1/p}}{\mu(2B)} \int_B |f(y)| \, d\mu(y) \]
\[ \leq \|f\|_{M_1^p(\mu)}. \]
Thus, (5) follows.

Analogously, the following inequality holds:

**Theorem 2.2.** Let \( 1 < q \leq p < \infty \). For positive \( \mu \)-measurable functions, inequality

\[
\| M_2 f \|_\mathcal{L}^p_q(6, \mu) \leq C_p \| f \|_\mathcal{L}^p_q(2, \mu)
\]

holds.

The proof of Theorem 2.2 being similar to that of Theorem 2.1, we skip the proof, where we use the \( L^q(\mu) \)-boundedness of \( M_2 \) established in [7].

Next, we prove a Hedberg type estimate.

**Theorem 2.3.** If \( 1 < p < \infty \) and \( 0 < \alpha < \frac{1}{p} \), then there exist \( C > 0 \) such that

\[
|I_\alpha f(x)| \leq CM_2 f(x)^{1-p\alpha} \| f \|^p_{\mathcal{L}^p_q(\mu)} \quad (x \in X)
\]

for all positive \( \mu \)-measurable functions.

**Proof.** Let \( x \in X \) be fixed. We define \( R_{k}(x) := \inf \{ R > 0 : \mu(B(x, 2R)) > 2^k \} \cap \{ \infty \} \). Then, we have

\[
|I_\alpha f(x)|
= \sum_{k=\infty}^{\infty} \lim_{\varepsilon \downarrow 0} \int_{B(x, R_k(x)) \setminus B(x, R_{k-1}(x))} \frac{|f(y)|}{\mu(B(x, 2d(x, y) + \varepsilon))^{1-\alpha}} \, d\mu(y)
= \sum_{k=\infty}^{\infty} \lim_{\varepsilon \downarrow 0} \int_{B(x, R_k(x)) \setminus B(x, R_{k-1}(x))} \frac{|f(y)|}{\mu(B(x, 2R_{k-1}(x) + \varepsilon))^{1-\alpha}} \, d\mu(y)
\leq \sum_{k=\infty}^{\infty} \lim_{\varepsilon \downarrow 0} \frac{1}{\mu(B(x, 2R_{k-1}(x) + \varepsilon))^{1-\alpha}} \int_{B(x, R_k(x)) \setminus B(x, R_{k-1}(x))} |f(y)| \, d\mu(y)
\leq \sum_{k=\infty}^{\infty} \lim_{\varepsilon \downarrow 0} \mu(B(x, 2R_{k-1}(x) + \varepsilon))^{1-\alpha} \int_{B(x, R_k(x))} |f(y)| \, d\mu(y).
\]

The condition \( R_{k-1}(x) < R_k(x) \) means \( 2^{k-1} < \mu(B(x, 2R_{k-1}(x) + \varepsilon)) \leq 2^k \) for each \( \varepsilon \in (0, R_k(x) - R_{k-1}(x)) \). Therefore

\[
|I_\alpha f(x)| \leq C \sum_{k=\infty}^{\infty} 2^{k\alpha} \min \left( M_2 f(x), 2^{-k/p} \| f \|_{\mathcal{L}^p_q(\mu)} \right)
\leq CM_2 f(x)^{1-p\alpha} \| f \|_{\mathcal{L}^p_q(\mu)}^{p\alpha}.
\]

Thus, the estimate is proved. \( \square \)
Now we prove Theorem 1.1.

Proof. For \(|I_{\alpha}f(x)| > \gamma\), Theorem 2.3 gives us

\[ M_2 f(x) > \left( \frac{\gamma}{C\|f\|_{M^p_1(\mu)}} \right)^{1/(1-p\alpha)}. \]

Hence, by applying Theorem 2.1, we obtain

\[ \mu \{ x \in B(a, r) : |I_{\alpha}f(x)| > \gamma \} \]
\[ \leq \mu \left\{ x \in B(a, r) : M_2 f(x) > \left( \frac{\gamma}{C\|f\|_{M^p_1(\mu)}} \right)^{1/(1-p\alpha)} \right\} \]
\[ \leq C\mu(B(a, r))^{1-1/p}\|f\|_{M^p_1(\mu)}\left( \frac{\|f\|_{M^p_1(\mu)}}{\gamma} \right)^{1/(1-p\alpha)} \]
\[ \leq C\mu(B(a, r))^{1-1/p}\frac{\|f\|_{M^p_1(\mu)}}{\gamma^{s/p}} \]
\[ \leq C\mu(B(a, r))^{1-1/p}\left( \frac{\|f\|_{M^p_1(\mu)}}{\gamma} \right)^{s/p}. \]

Thus, the proof is complete. \(\square\)

Theorem 1.2 can be proven in a similar way using Theorem 2.3.

Acknowledgments. The first author is supported by Fundamental Research Program 2012 by Directorate General of Higher Education, Ministry of Education and Culture, Indonesia. The second author is supported financially by Grant-in-Aid for Young Scientists (B), No. 21740104, Japan Society for the Promotion of Science. This research project is supported by the GCOE program of Kyoto University.

References


Idha SIHWANINGRUM
Faculty of Sciences and Engineering
Jenderal Soedirman University
Purwokerto, 53122 Indonesia
email: idha.sihwaningrum@unsoed.ac.id

* Corresponding author:
YOSHIHIRO SAWANO
Department of Mathematics and Information Sciences
Tokyo Metropolitan University
Tokyo 192-0397, Japan
email: ysawano@tmu.ac.jp