WEAK TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED NON-HOMOGENEOUS MORREY SPACES

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Abstract
We obtain weak type \((1, q)\) inequalities for fractional integral operators on generalized non-homogeneous Morrey spaces. The proofs use some properties of maximal operators. Our results are closely related to the strong type inequalities in [13, 14, 15].

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1 Introduction
The works of Nazarov et al. [10], Tolsa [17], and Verdera [18] reveal some important ideas of the spaces of non-homogeneous type. By a non-homogeneous space we mean a (metric) measure space — here we consider only \(\mathbb{R}^d\) — equipped with a Borel measure \(\mu\) satisfying the growth condition of order \(n\) with \(0 < n \leq d\), that is there exists a constant \(C > 0\) such that

\[
\mu(B(a, r)) \leq C r^n
\]

for every ball \(B(a, r)\) centered at \(a \in \mathbb{R}^d\) with radius \(r > 0\). The growth condition replaces the doubling condition:

\[
\mu(B(a, 2r)) \leq C\mu(B(a, r))
\]

which plays an important role in the spaces of homogeneous type.

In the setting of non-homogeneous spaces described above, we define the fractional integral operator \(I_\alpha\) \((0 < \alpha < n \leq d)\) by the formula

\[
I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y)
\]

for any suitable function \(f\) on \(\mathbb{R}^d\). Note that if \(n = d\) and \(\mu\) is the usual Lebesgue measure on \(\mathbb{R}^d\), then \(I_\alpha\) is the classical fractional integral operator introduced by Hardy and Littlewood [5, 6] and Sobolev [16]. The classical
fractional integral operator $I_\alpha$ is known to be bounded from the usual Lebesgue space $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ for $1 < p < \frac{d}{\alpha}$. This result has been extended in many ways — see for examples [4, 8, 11] and the references therein.

For $p = 1$, we have a weak type inequality for $I_\alpha$ and on non-homogeneous Lebesgue spaces such an inequality has been studied, among others, by García-Cuerva, Gatto, and Martell in [2, 3]. One of their results is the following theorem. (Here and after, we denote by $C$ a positive constant which may have different values from line to line.)

**Theorem 1.1.** [2, 3] If $\frac{1}{q} = 1 - \frac{\alpha}{n}$, then for any function $f \in L^1(\mu)$ we have

$$\mu \{ x \in \mathbb{R}^d : |I_\alpha f(x)| > \gamma \} \leq C \left( \frac{\|f\|_{L^1(\mu)}}{\gamma} \right)^q, \quad \gamma > 0.$$  

The proof of Theorem 1.1 uses the weak type inequality for the maximal operator

$$Mf(x) := \sup_{r > 0} \frac{1}{r^n} \int_{B(x, r)} |f(y)| \, d\mu(y).$$

In this paper, we shall prove the weak type inequality for $I_\alpha$ on generalized non-homogeneous Morrey spaces (which we shall define later). The proof will employ the following inequality for the maximal operator $M$.

**Theorem 1.2.** [2, 12] For any positive weight $w$ on $\mathbb{R}^d$ and any function $f \in L^1_{\text{loc}}(\mu)$, we have

$$\int_{\{ x \in \mathbb{R}^d : Mf(x) > \gamma \}} w(x) \, d\mu(x) \leq \frac{C}{\gamma} \int_{\mathbb{R}^d} |f(x)| Mw(x) \, d\mu(x), \quad \gamma > 0.$$  

Our main results are presented as Theorems 2.2 and 2.3 in the next section. Related results can be found in [13, 14, 15].

2 Main Results

For $1 \leq p < \infty$ and a suitable function $\phi : (0, \infty) \to (0, \infty)$, we define the generalized non-homogeneous Morrey space $M^{p,\phi}(\mu) = M^{p,\phi}(\mathbb{R}^d, \mu)$ to be the space of all functions $f \in L^p_{\text{loc}}(\mu)$ for which

$$\|f\|_{M^{p,\phi}(\mu)} := \sup_{B = B(a, r)} \frac{1}{\phi(r)} \left( \frac{1}{r^n} \int_B |f(x)|^p \, d\mu(x) \right)^{1/p} < \infty.$$  

(We refer the reader to [1] for the definition of analogous spaces in the homogeneous case.) Throughout this paper, we will always assume that $\phi$ is an almost decreasing function, that is there exists a constant $C > 0$ such that $\phi(t) \leq C \phi(s)$ whenever $s < t$.

Our first theorem is closely related to Theorem 3.3 in [14].
Theorem 2.1. If the function $\phi : (0, \infty) \rightarrow (0, \infty)$ satisfies
\[ \int_r^\infty \frac{\phi(t)}{t} \, dt \leq C\phi(r), \quad r > 0, \]
then for any function $f \in M^1\phi(\mu)$ and any ball $B(a, r) \subseteq \mathbb{R}^d$ we have
\[ \mu\{x \in B(a, r) : Mf(x) > \gamma\} \leq \frac{C}{\gamma} r^n \phi(r) \|f\|_{M^1\phi(\mu)}, \quad \gamma > 0. \]

Proof. For any function $f \in M^1\phi(\mu)$ and the characteristic function $\chi_{B(a, r)}$, we observe that
\[ \int_{\mathbb{R}^d} |f(x)| M\chi_{B(a, r)}(x) \, d\mu \leq \int_{B(a, 2r)} |f(x)| M\chi_{B(a, r)}(x) \, d\mu \]
\[ + \sum_{k=1}^{\infty} \int_{B(a, 2^{k+1}r) \setminus B(a, 2^kr)} |f(x)| M\chi_{B(a, r)}(x) \, d\mu. \]
Since $\mu$ satisfies the growth condition (1), we have $M\chi_{B(a, r)}(x) \leq C$ and $M\chi_{B(a, r)}(x) \leq C 2^{-kn}$ whenever $x \in B(a, 2^{k+1}r) \setminus B(a, 2^kr)$ (where $k = 1, 2, 3, \ldots$). Now, as $\phi$ is almost increasing, we have
\[ \phi(2^{k+1}r) \leq C \int_{2^kr}^{2^{k+1}r} \frac{\phi(t)}{t} \, dt \]
for $k = 1, 2, 3, \ldots$. Consequently,
\[ \int_{\mathbb{R}^d} |f(x)| M\chi_{B(a, r)}(x) \, d\mu \]
\[ \leq C \left( \int_{B(a, 2r)} |f(x)| \, d\mu + \sum_{k=1}^{\infty} \int_{B(a, 2^{k+1}r) \setminus B(a, 2^kr)} |f(x)| 2^{-kn} \, d\mu \right) \]
\[ \leq C \left( (2r)^n \phi(2r) \|f\|_{M^1\phi(\mu)} + \sum_{k=1}^{\infty} 2^{-kn} (2^{k+1}r)^n \phi(2^{k+1}r) \|f\|_{M^1\phi(\mu)} \right) \]
\[ = Cr^n \|f\|_{M^1\phi(\mu)} \sum_{k=0}^{\infty} \phi(2^{k+1}r) \]
\[ \leq Cr^n \|f\|_{M^1\phi(\mu)} \sum_{k=0}^{\infty} \int_{2^kr}^{2^{k+1}r} \frac{\phi(t)}{t} \, dt \]
\[ \leq Cr^n \|f\|_{M^1\phi(\mu)} \int_r^{\infty} \frac{\phi(t)}{t} \, dt \]
\[ \leq Cr^n \phi(r) \|f\|_{M^1\phi(\mu)}. \]
Next, by applying Theorem 1.2, we find that for \( \gamma > 0 \),
\[
\mu\{x \in B(a, r) : Mf(x) > \gamma\} = \int_{\{x \in B(a, r) : Mf(x) > \gamma\}} \chi_{B(a, r)}(x) \, d\mu
\leq \frac{C}{\gamma} \int_{\mathbb{R}^d} |f(x)| \chi_{B(a, r)}(x) \, d\mu
\leq \frac{C}{\gamma} r^n \phi(r) \|f\|_M^{1, \phi(\mu)},
\]
as desired.

Theorem 2.1 enables us to obtain an inequality in which the fractional integral operator is controlled by the maximal operator. The classical setting of this inequality is available in [7].

**Theorem 2.2.** Suppose that, for some \( 0 \leq \lambda < n - \alpha \), we have
\[
\int_r^\infty t^{\alpha - 1} \phi(t) \, dt \leq C r^{\lambda + \alpha - n}, \quad r > 0.
\]
Then, for any function \( f \in \mathcal{M}^{1, \phi(\mu)} \), we have
\[
|I_\alpha f(x)| \leq C [Mf(x)]^{1 - \frac{\alpha}{n - \alpha}} \|f\|_M^{\alpha/(n - \lambda)}, \quad x \in \mathbb{R}^d.
\]

**Proof.** Let \( f \in \mathcal{M}^{1, \phi(\mu)} \) and \( x \in \mathbb{R}^d \). For every \( r > 0 \), we have
\[
|I_\alpha f(x)| \leq \int_{|x - y| < r} \frac{|f(y)|}{|x - y|^{n - \alpha}} \, d\mu(y) + \int_{|x - y| \geq r} \frac{|f(y)|}{|x - y|^{n - \alpha}} \, d\mu(y)
=: A + B.
\]
Observe that for the first term we obtain
\[
A = \int_{|x - y| < r} \frac{|f(y)|}{|x - y|^{n - \alpha}} \, d\mu(y)
\leq \sum_{j = -\infty}^{-1} \int_{2^j r < |x - y| \leq 2^{j + 1} r} \frac{|f(y)|}{|x - y|^{n - \alpha}} \, d\mu(y)
\leq \sum_{j = -\infty}^{-1} \frac{1}{(2^j r)^{n - \alpha}} \int_{|x - y| \leq 2^{j + 1} r} |f(y)| \, d\mu(y)
\leq \sum_{j = -\infty}^{-1} 2^n (2^j r)^n \frac{1}{(2^j + 1)^n} \int_{B(x, 2^j + 1 r)} |f(y)| \, d\mu(y)
\leq 2^n r^\alpha Mf(x) \sum_{j = -\infty}^{-1} 2^{j\alpha}
\leq C r^\alpha Mf(x).
\]
Meanwhile, for the second term, we have the following estimate:

\[
B = \int_{|x-y| \geq r} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, d\mu(y)
\]
\[
= \sum_{j=0}^{\infty} \int_{2^j r < |x-y| \leq 2^{j+1} r} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, d\mu(y)
\]
\[
\leq \sum_{j=0}^{\infty} \alpha \left( \frac{2^j r}{2^{j+1} r} \right)^{n-\alpha} \int_{|x-y| \leq 2^{j+1} r} |f(y)| \, d\mu(y)
\]
\[
= \sum_{j=0}^{\infty} 2^n (2^j r)^{\alpha} \int_{B(x, 2^{j+1} r)} |f(y)| \, d\mu(y)
\]
\[
\leq C \|f\|_{M^{1, \alpha}(\mu)} \sum_{j=0}^{\infty} (2^j r)^{\alpha} \phi(2^{j+1} r).
\]

As \(\phi\) is almost decreasing, we observe that for \(j = 0, 1, 2, \ldots\),

\[
(2^j r)^{\alpha} \phi(2^{j+1} r) \leq C \int_{2^j r}^{2^{j+1} r} t^{\alpha-1} \phi(t) \, dt.
\]

This last inequality and our assumption then lead us to

\[
B \leq C \|f\|_{M^{1, \alpha}(\mu)} \sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} t^{\alpha-1} \phi(t) \, dt
\]
\[
\leq C \|f\|_{M^{1, \alpha}(\mu)} \int_{r}^{\infty} t^{\alpha-1} \phi(t) \, dt
\]
\[
\leq C r^{\lambda+\alpha-n} \|f\|_{M^{1, \alpha}(\mu)}.
\]

Now, by choosing

\[
r = \left( \frac{M f(x)}{\|f\|_{M^{1, \alpha}(\mu)}} \right)^{\frac{1}{n-\alpha}}
\]

we obtain

\[
|I_- f(x)| \leq C r^{\alpha} \left( M f(x) + r^{\lambda-n} \|f\|_{M^{1, \alpha}(\mu)} \right)
\]
\[
\leq C [M f(x)]^{1-\frac{\alpha}{n-\alpha}} \|f\|_{M^{1, \alpha}(\mu)}^{(n-\lambda)/n}.
\]

This completes the proof. \(\square\)

Now, with the use of Theorem 2.1 and 2.2, we obtain the following weak type \((1, q)\) inequality for \(I_-\). Our result is analogous to that of [9] in homogeneous setting.
Theorem 2.3. If $\frac{1}{q} = 1 - \frac{\alpha}{n - \lambda}$ and $\phi$ satisfies the conditions in Theorems 2.1 and 2.2, then for any function $f \in M_{1,\phi}(\mu)$ and any ball $B(a,r) \subseteq \mathbb{R}^d$ we have

$$\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \leq C r^n \phi(r) \left( \frac{\|f\|_{M_{1,\phi}(\mu)}}{\gamma} \right)^q, \quad \gamma > 0.$$ 

Proof. If $|I_\alpha f(x)| > \gamma$, then Theorem 2.2 gives us

$$Mf(x) > \left( \frac{\gamma}{C\|f\|_{M_{1,\phi}(\mu)}^{\alpha/(n-\lambda)}} \right)^{\frac{n-\lambda}{n-\lambda-n}} = \left( \frac{\gamma}{C\|f\|_{M_{1,\phi}(\mu)}^{\alpha/(n-\lambda)}} \right)^q.$$

Furthermore, by using Theorem 2.1, we get

$$\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \leq \mu \left\{ x \in B(a,r) : Mf(x) > \left( \frac{\gamma}{C\|f\|_{M_{1,\phi}(\mu)}^{\alpha/(n-\lambda)}} \right)^q \right\}$$

$$\leq C r^n \phi(r) \|f\|_{M_{1,\phi}(\mu)} \left( \frac{\|f\|_{M_{1,\phi}(\mu)}^{\alpha/(n-\lambda)}}{\gamma} \right)^q$$

$$= C r^n \phi(r) \left( \frac{\|f\|_{M_{1,\phi}(\mu)}^{1/q + \alpha/(n-\lambda)}}{\gamma} \right)^q$$

$$= C r^n \phi(r) \left( \frac{\|f\|_{M_{1,\phi}(\mu)}}{\gamma} \right)^q,$$

which is the desired inequality.

Remark. Note that when $\phi(t) = t^{\lambda-n}$ with $0 \leq \lambda < n - \alpha$, we get $M_{1,\phi}(\mu) = L^{1,\lambda}((\mu))$, the usual Morrey spaces of non-homogeneous type. In this case, the above inequality reduces to

$$\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \leq C r^\lambda \left( \frac{\|f\|_{L^{1,\lambda}((\mu))}}{\gamma} \right)^q, \quad \gamma > 0.$$ 

Furthermore, if $\lambda = 0$, then $L^{1,0}((\mu)) = L^1((\mu))$ and for $\frac{1}{q} = 1 - \frac{\alpha}{n}$ we obtain

$$\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \leq C \left( \frac{\|f\|_{L^1((\mu))}}{\gamma} \right)^q, \quad \gamma > 0.$$ 

Since the inequality holds for any ball $B(a,r)$, we deduce that

$$\mu\{x \in \mathbb{R}^d : |I_\alpha f(x)| > \gamma\} \leq C \left( \frac{\|f\|_{L^1((\mu))}}{\gamma} \right)^q, \quad \gamma > 0.$$ 

as in Theorem 1.1.

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