The Boundedness of Bessel-Riesz Operators
On Morrey Spaces

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Abstract. In this paper, we shall discuss about Bessel-Riesz operators. Kurata \textit{et. al} \cite{6} have investigated their boundedness on generalized Morrey spaces with weight. The boundedness of these operators on Lebesgue spaces and Morrey spaces will be reproved using a different approach. Moreover, we also find the norm of the operators are bounded by the norm of the kernels.

Keywords: Bessel-Riesz operators, Hardy-Littlewood maximal operator, Morrey spaces.

INTRODUCTION

Let \(0 < \gamma, 0 < \alpha < n\) and define

\[ I_{\alpha, \gamma} f(x) := \int_{\mathbb{R}^n} K_{\alpha, \gamma}(x - y) f(y) \, dy, \text{ where } K_{\alpha, \gamma}(x) := \frac{|x|^{\alpha - n}}{(1 + |x|)^{\gamma}}, x \in \mathbb{R}^n \]

for every \(f \in L^p(\mathbb{R}^n), \ p \geq 1\). Here, \(K_{\alpha, \gamma}\) can be viewed as multiple of two kernels, \(K_{\alpha, \gamma}(x) = J_{\gamma}(x) \cdot K_{\alpha}(x)\) for every \(x \in \mathbb{R}^n\). In \cite{8}, \(J_{\gamma}\) and \(K_{\alpha}\) are known as \textit{Bessel kernel} and \textit{Riesz kernel}. So, \(K_{\alpha, \gamma}\) is called \textit{Bessel-Riesz kernel} and \(I_{\alpha, \gamma}\) is called \textit{Bessel-Riesz operators}.

For \(\gamma = 0\), we have \(I_{0, 0} = I_0\) (is called \textit{fractional integral operators} or \textit{Riesz potentials} \cite{9}). Studies about \(I_\alpha\) were started since 1920’s. Hardy-Littlewood \cite{4, 5} and Sobolev \cite{8} proved the boundedness of \(I_\alpha\) on Lebesgue spaces through the inequality \(\|I_\alpha f\|_{L^q} \leq C_p \|f\|_{L^p}\), for every \(f \in L^p(\mathbb{R}^n), 1 < p < \frac{n}{\alpha}\), and \(\frac{1}{q} = \frac{1}{p} - \frac{n}{\alpha}\).

For \(1 \leq p < q\), the (classical) Morrey space \(L^{p,q}(\mathbb{R}^n)\) is defined by

\[ L^{p,q}(\mathbb{R}^n) := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,q}} < \infty \right\}, \]

where \(\|f\|_{L^{p,q}} := \sup_{r>0, 0<\varepsilon<\varepsilon} r^\alpha (\int_{|x|<\varepsilon, |x|>\varepsilon} |f(x)|^p \, dx)^{1/p}\). We have an inclusion property for Morrey spaces \(L^q(\mathbb{R}^n) = L^{q,q}(\mathbb{R}^n) \subseteq L^{p,q}(\mathbb{R}^n) \subseteq L^{1,q}(\mathbb{R}^n)\).

On Morrey spaces, Spanne \cite{7} has shown that \(I_\alpha\) is bounded form \(L^{p_1;q_1}(\mathbb{R}^n)\) to \(L^{p_2;q_2}(\mathbb{R}^n)\) for \(1 < p_1 < q_1 < \frac{n}{\alpha}\), \(\frac{1}{p_2} = \frac{1}{p_1} - \frac{q_1}{q_2}\), and \(\frac{1}{q_2} = \frac{1}{q_1} - \frac{n}{\alpha}\). Furthermore, Adams \cite{1} and Chiarenza-Frasca \cite{2} obtained a stronger result.

Theorem 1 \cite{Adams, Chiarenza-Frasca} \textit{If} \(0 < \alpha < n\) \textit{then we have}

\[ \|I_\alpha f\|_{L^{p_2;q_2}} \leq C_{p_1,q_1} \|f\|_{L^{p_1,q_1}}, \]

\textit{for every} \(f \in L^{p_1,q_1}(\mathbb{R}^n)\) \textit{where} \(1 < p_1 < q_1 < \frac{n}{\alpha}\), \(\frac{1}{p_2} = \frac{1}{p_1} \left(1 - \frac{q_1}{n}\right)\) \textit{and} \(\frac{1}{q_2} = \frac{1}{q_1} - \frac{n}{\alpha}\).

Meanwhile, we have \(\|I_{\alpha, \gamma} f(x)\| \leq \|I_\alpha f(x)\|\), \textit{for every} \(f \in L^p_{\text{loc}}(\mathbb{R}^n)\). Using this inequality, \(I_{\alpha, \gamma}\) is bounded on these spaces. In 1999, Kurata \textit{et. al} \cite{6} have proved that \(W \cdot I_{\alpha, \gamma}\) is bounded on generalized Morrey spaces where \(W\) is a multiplication operator. Here, we shall discuss about the boundedness of \(I_{\alpha, \gamma}\) on Lebesgue spaces and Morrey spaces. We shall see the influence of \(K_{\alpha, \gamma}\) for the boundedness of \(I_{\alpha, \gamma}\).
PRELIMINARY RESULTS

We can see that the kernel of Bessel-Riesz vanishes faster at infinity than that of the fractional integral operator. From this fact, we can show that the kernel of Bessel-Riesz is a member of some Lebesgue spaces. We begin with the following lemma.

**Lemma 2** If $b > a > 0$ then $\sum_{k \in \mathbb{Z}} \frac{\left( \psi(u) \right)^a}{(1+u^2)^b} < \infty$, for every $u > 1$ and $R > 0$.

**Proof.** Let $b > a > 0$, so that $b - a > 0$. We write $\sum_{k \in \mathbb{Z}} \frac{\left( \psi(u) \right)^a}{(1+u^2)^b} = \sum_{k=-\infty}^{\infty} \frac{\left( \psi(u) \right)^a}{(1+u^2)^b}$. Next, we estimate $\sum_{k=-\infty}^{\infty} \frac{\left( \psi(u) \right)^a}{(1+u^2)^b} < \infty$ and $\sum_{k=-0}^{\infty} \frac{\left( \psi(u) \right)^a}{(1+u^2)^b} < \infty$. Hence, we obtain $\sum_{k \in \mathbb{Z}} \frac{\left( \psi(u) \right)^a}{(1+u^2)^b} < \infty$. □

Lemma 2 is useful to prove the membership of $K_{n,0}$ in some Lebesgue spaces.

**Theorem 3** If $0 < \alpha < n$ and $0 < \gamma$ then $K_{n,\gamma} \in L^1\left( \mathbb{R}^n \right)$ and $\left\| K_{n,\gamma} \right\|_{L^1} \sim \left( \sum_{k \in \mathbb{Z}} \frac{\left( \psi(u) \right)^a}{(1+u^2)^b} \right)^{\frac{1}{\gamma}}$, for $\frac{n}{\gamma} - a < t < \frac{n}{\gamma} - a$.

**Proof.** Suppose $\frac{n}{\gamma} - a < t < \frac{n}{\gamma} - a$ where $0 < \gamma$, $0 < \alpha < n$, so that $(a-n)t + n > 0$. For arbitrary $R > 0$, write

$$
\int_{\mathbb{R}^n} |K_{n,\gamma}(y)|^{\gamma} \, dy = \int_{\mathbb{R}^n} \left( \frac{1}{(1+|y|)^\gamma} \right) \, dy = C_1 \int_{\mathbb{R}^n} \left( \frac{1}{1+|y|} \right) \, dy = C_1 \sum_{k \in \mathbb{Z}} \int_{2^k R < |y| < 2^{k+1} R} \left( \frac{1}{1+|y|} \right) \, dy, \quad C_1 > 0.
$$

We obtain $\int_{\mathbb{R}^n} |K_{n,\gamma}(y)|^{\gamma} \, dy \leq C_1 \sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^\gamma} \int_{2^k R < |y| < 2^{k+1} R} \left( \frac{1}{1+|y|} \right) \, dy = C_2 \sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^\gamma} \int_{2^k R < |y| < 2^{k+1} R} \left( \frac{1}{1+|y|} \right) \, dy = C_2 \sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^\gamma} \int_{2^k R < |y| < 2^{k+1} R} \left( \frac{1}{1+|y|} \right) \, dy$, and $\sum_{k \in \mathbb{Z}} |K_{n,\gamma}(y)|^{\gamma} \, dy \leq \sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^\gamma} \int_{2^k R < |y| < 2^{k+1} R} \left( \frac{1}{1+|y|} \right) \, dy = C_3 \sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^\gamma} \int_{2^k R < |y| < 2^{k+1} R} \left( \frac{1}{1+|y|} \right) \, dy$, therefore $\int_{\mathbb{R}^n} |K_{n,\gamma}(y)|^{\gamma} \, dy \leq C_2 \sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^\gamma} \int_{2^k R < |y| < 2^{k+1} R} \left( \frac{1}{1+|y|} \right) \, dy$. Thus, we get $\sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^\gamma} \int_{2^k R < |y| < 2^{k+1} R} \left( \frac{1}{1+|y|} \right) \, dy < \infty$. Hence $K_{n,\gamma} \in L^1\left( \mathbb{R}^n \right)$. □

In this study, the membership of $K_{n,\gamma}$ in Lebesgue spaces is an important result. With the result, we can use Young inequality [3] to prove the boundedness of $I_{n,\gamma}$ on Lebesgue spaces.

**Theorem 4** (Young’s inequality) Let $1 \leq p, q, t \leq \infty$ satisfy $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{t}$, then we have $\|g * f\|_{L^t} \leq \|g\|_{L^q} \|f\|_{L^p}$ for every $g \in L^q\left( \mathbb{R}^n \right)$, $f \in L^p\left( \mathbb{R}^n \right)$.

**Corollary 5** For $0 < \alpha < n$, $\gamma > 0$, we have $\|I_{n,\gamma}f\|_{L^t} \leq \|K_{n,\gamma}\|_{L^1} \|f\|_{L^p}$ for every $f \in L^p\left( \mathbb{R}^n \right)$ where $1 \leq p < t$, $\frac{n}{\gamma} - a < t < \frac{n}{\gamma} - a$, $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{t}$.

By the above corollary, we can say that $I_{n,\gamma}$ is bounded from $L^p\left( \mathbb{R}^n \right)$ to $L^s\left( \mathbb{R}^n \right)$. Moreover, norm of kernel Bessel-Riesz dominates norm of $I_{n,\gamma}f$. Consequently in Lebesgue spaces, we obtain $\|I_{n,\gamma}\| \leq \|K_{n,\gamma}\|_{L^1}$.

We shall extend the boundedness of $I_{n,\gamma}$ on Morrey spaces, but Young’s inequality is not available on Morrey spaces. Using the Hardy-Littlewood maximal operator $M$, the boundedness of $I_{n,\gamma}$ can be reproved on Lebesgue spaces and Morrey spaces. The operator $M$ is defined by

$$Mf(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| \, dy, x \in \mathbb{R}^n,$$

for every $f \in L^p_{loc}\left( \mathbb{R}^n \right)$ where $|B|$ denotes Lebesgue measure of ball $B = B(a, r)$ (centered at $a \in \mathbb{R}^n$ with radius $r > 0$). The supremum is taken over all open balls in $\mathbb{R}^n$. It is well known that the operator $M$ is bounded on Lebesgue spaces ($L^p\left( \mathbb{R}^n \right)$, $p > 1$) [9, 10] and Morrey spaces [2].
MAIN RESULTS

In this section, we are going to discuss about the boundedness of the Bessel-Riesz operators on Morrey spaces. In the previous section, we have $K_{α,γ} \in L^i(\mathbb{R}^n)$ where $\frac{n}{n+γ} < i < \frac{n}{n−α}$ and the inclusion property of Morrey spaces, so $K_{α,γ} \in L^i(\mathbb{R}^n)$ where $1 ≤ s ≤ t$. Accordingly, we have the following theorem.

**Theorem 6** Let $0 < α < n, 0 < γ$, then we have

$$\|I_{α,γ}f\|_{L^{p,q}_{p,q}} \leq C_{p,q,γ} \frac{\|K_{α,γ}\|_{L^i_{p,q}} \|f\|_{L^i_{p,q}}}{1}.$$ 

for every $f \in L^{p,q}_{p,q}(\mathbb{R}^n)$ where $1 < p < q, 1 ≤ s ≤ t$, where $\frac{n}{n+γ} < i < \frac{n}{n−α}, \frac{1}{p} = \frac{1}{p_1} − \frac{1}{q_1},$ and $\frac{1}{q} = \frac{1}{q_1} − \frac{1}{p}.$

**Proof.** Suppose $0 < α < n, 0 < γ$ and take $\frac{n}{n+γ} < i < \frac{n}{n−α}, 1 ≤ s ≤ t.$ Let $f \in L^{p,q}_{p,q}(\mathbb{R}^n), 1 < p < q, 1 ≤ s ≤ t.$ For every $x \in \mathbb{R}^n$, write $I_{α,γ}f(x) := I_1(x) + I_2(x)$ where $I_1(x) := \int_{|x−y| < R} \frac{|x−y|^{−α−γ}}{|y|^s}f(y)dy$ and $I_2(x) := \int_{|x−y| ≥ R} \frac{|x−y|^{−α−γ}}{|y|^s}f(y)dy, R > 0.$ To estimate $I_1$ and $I_2$, we use dyadic decomposition. Now, estimate $I_1$

$$|I_1(x)| \leq C_1 \sum_{k=1}^{∞} \frac{(2^k R)^{α−n}}{(1 + 2^k R)^{2γ}} \int_{2^k R ≤ |x−y| ≤ 2^{k+1} R} |f(y)|dy \leq C_2 Mf(x) \sum_{k=1}^{∞} \frac{(2^k R)^{α−n+γ}}{(1 + 2^k R)^{2γ}}.$$ 

By using Hölder’s inequality, we get

$$|I_1(x)| \leq C_3 Mf(x) \left( \sum_{k=1}^{∞} \frac{(2^k R)^{α−n+γ+q}}{(1 + 2^k R)^{2γ}} \right)^{1/s} \left( \sum_{k=1}^{∞} \frac{(2^k R)^{α}}{(1 + 2^k R)^{2γ}} \right)^{1/s} \leq C_4 Mf(x) \frac{\int_{|x−y| < R} K_{α,γ}(x−y)dy}{R^{n/(s−1)}} R^{n/s} \leq C_4 \frac{\|K_{α,γ}\|_{L^i_{p,q}} \|Mf\|_{L^i_{p,q}}}{R^{n/(s−1)}} R^{n/s}.$$ 

Hölder’s inequality is used again to estimate $I_2$:

$$|I_2(x)| \leq C_5 \sum_{k=0}^{∞} \frac{(2^k R)^{α−n}}{(1 + 2^k R)^{2γ}} \int_{2^k R ≤ |x−y| ≤ 2^{k+1} R} |f(y)|dy \leq C_5 \sum_{k=0}^{∞} \frac{(2^k R)^{α−n−q}}{(1 + 2^k R)^{2γ}} \left( \int_{2^k R ≤ |x−y| ≤ 2^{k+1} R} |f(y)|^{p_1} \right)^{1/p_1} \left( \frac{2^k R)^{α}}{(1 + 2^k R)^{2γ}} \right)^{1/s}.$$ 

Next, we write

$$|I_2(x)| \leq C_6 \|f\|_{L^{p,q}_{p,q}} \sum_{k=0}^{∞} \frac{(2^k R)^{α−n−q} \int_{2^k R ≤ |x−y| ≤ 2^{k+1} R} dy}{(1 + 2^k R)^{2γ}} \leq C_6 \|f\|_{L^{p,q}_{p,q}} \sum_{k=0}^{∞} \frac{\int_{2^k R ≤ |x−y| ≤ 2^{k+1} R} |y−1|^{−α−n−q} |y|^{−γ}dy}{R^{n/(s−1)}}.$$ 

and we obtain $|I_2(x)| \leq C_6 \|f\|_{L^{p,q}_{p,q}} \|K_{α,γ}\|_{L^i_{p,q}} \sum_{k=0}^{∞} \frac{(2^k R)^{n/(s−1)}}{R^{n/(s−1)}} \leq C_7 \|K_{α,γ}\|_{L^i_{p,q}} \|f\|_{L^{p,q}_{p,q}} R^{n/(s−1−n)}. Summing the two estimates, we get $|I_{α,γ}f(x)| \leq C \|K_{α,γ}\|_{L^i_{p,q}} \|f\|_{L^{p,q}_{p,q}} R^{n/(s−1)}. We get $|I_{α,γ}f(x)| \leq C \|K_{α,γ}\|_{L^i_{p,q}} \|f\|_{L^{p,q}_{p,q}} (Mf(x))^{1−n/|r|}.$ Define $\frac{1}{p_1} := \frac{|r|−n}{|r|}$ and $\frac{1}{q_1} := \frac{1}{q} − \frac{1}{p}.$ For arbitrary $r > 0,$ we have

$$\left( \int_{|x| ≤ r} |I_{α,γ}f(x)|^{p_1} \right)^{1/p_1} \leq C \left( \int_{|x| ≤ r} |Mf(x)|^{q_1} \right)^{1/q_1}.$$ 

Divide by $r^{n/(s−1−n)}$ and take supremum to get

$$\|I_{α,γ}f\|_{L^{p,q}_{p,q}} = \sup_{r > 0} \left( \int_{|x| ≤ r} |I_{α,γ}f(x)|^{p_1} \right)^{1/p_1} \leq C \|K_{α,γ}\|_{L^i_{p,q}} \|f\|_{L^{p,q}_{p,q}}^{1−n/|r|}.$$ 

Therefore, $f \in L^i_{p,q}$ and for arbitrary $r > 0,$ we have

$$\left( \int_{|x| ≤ r} |I_{α,γ}f(x)|^{p_1} \right)^{1/p_1} \leq C \left( \int_{|x| ≤ r} |Mf(x)|^{q_1} \right)^{1/q_1}.$$ 

Divide by $r^{n/(s−1−n)}$ and take supremum to get

$$\|I_{α,γ}f\|_{L^{p,q}_{p,q}} = \sup_{r > 0} \left( \int_{|x| ≤ r} |I_{α,γ}f(x)|^{p_1} \right)^{1/p_1} \leq C \|K_{α,γ}\|_{L^i_{p,q}} \|f\|_{L^{p,q}_{p,q}}^{1−n/|r|}.\]
Using the boundedness of $M$ on Morrey spaces (Chiarenza-Frasca’s Theorem [2]), we obtain an inequality
\[ \|I_{\alpha,\gamma}f\|_{L^{p,q}_2} \leq C_{p,q_1} \|K_{\alpha,\gamma}\|_{L^p_t} \|f\|_{L^{p_1,q_1}}. \]

By Theorem 6 and the inclusion property of Morrey spaces, for $1 \leq s \leq t$, we have
\[ \|I_{\alpha,\gamma}f\|_{L^{p,q}_2} \leq C_{p,q_1} \|K_{\alpha,\gamma}\|_{L^p_s} \|f\|_{L^{p_1,q_1}} \leq C_{p,q_1} \|K_{\alpha,\gamma}\|_{L^t_t} \|f\|_{L^{p_1,q_1}}. \]
where \( \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha} \). We also obtain \( \frac{p_2}{q_2} = \frac{p_1}{q_1} \). It is similar with Chiarenza-Frasca’s result for the boundedness of fractional integral operators on Morrey spaces.

**CONCLUDING REMARK**

From the result of this study, we have seen that the norm of the Bessel-Riesz kernel dominates the norm of \( I_{\alpha,\gamma}f \) for every \( f \) in Morrey space \( L^{p,q}(\mathbb{R}^n) \) (\( p \) and \( q \) are suitable numbers). Moreover, using \( K_{\alpha,\gamma} \in L^{s,t}(\mathbb{R}^n) \), \( 1 \leq s < t \), \( \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha} \), the norm of the Bessel-Riesz kernel closer to the norm of \( I_{\alpha,\gamma}f \) than using \( K_{\alpha,\gamma} \in L^t(\mathbb{R}^n) \). In the future, we shall continue this study to prove the boundedness of generalized Bessel-Riesz operators on Morrey spaces and generalized Morrey spaces.

**REFERENCES**