Inclusion Properties of Generalized Morrey Spaces

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Abstract

This paper discusses the structure of Morrey spaces, weak Morrey spaces, generalized Morrey spaces, and generalized weak Morrey spaces. Some necessary and sufficient conditions for the inclusion property of these spaces are obtained through a norm estimate for the characteristic functions of balls.

Keywords: Morrey spaces, weak Morrey spaces, generalized Morrey spaces, fractional integral operators.

1 Introduction

Let $L^p_{\text{loc}}(\mathbb{R}^d)$ denote the space of all $p$-locally integrable functions on $\mathbb{R}^d$. For $1 \leq p \leq q < \infty$, we define the Morrey space $\mathcal{M}_q^p(\mathbb{R}^d)$ by

$$\mathcal{M}_q^p(\mathbb{R}^d) := \{ f \in L^p_{\text{loc}}(\mathbb{R}^d) : \| f \|_{\mathcal{M}_q^p} < \infty \},$$

where $\| \cdot \|_{\mathcal{M}_q^p}$ is given by

$$\| f \|_{\mathcal{M}_q^p} := \sup_{a \in \mathbb{R}^d, r > 0} |B(a, r)|^{-\frac{1}{q}} \left( \frac{1}{|B(a, r)|} \int_{B(a,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Here, $B(a, r)$ denotes the open ball in $\mathbb{R}^d$ centered at $a$ with radius $r$, and $|B(a, r)|$ denotes its Lebesgue measure. One might observe that $\| \cdot \|_{\mathcal{M}_q^p}$ defines a norm on $\mathcal{M}_q^p$. Also note that if $p = q$, then $\mathcal{M}_q^p(\mathbb{R}^d) = L^p(\mathbb{R}^d)$. Thus, $\mathcal{M}_q^p(\mathbb{R}^d)$ can be viewed as a generalization of the Lebesgue space $L^p(\mathbb{R}^d)$.

Morrey spaces were first introduced by C.B. Morrey in 1938 [13]. A few decades later some researchers were interested in studying the boundedness of various operators on these spaces. In this direction, one of the important results on Morrey spaces is the boundedness of the fractional integral operator $I_\alpha$, which is defined for $0 < \alpha < d$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} \, dy,$$

for any locally integrable function $f$ on $\mathbb{R}^d$. 

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Theorem 1.1. [1, 3] Let \(1 < p \leq q < \frac{d}{\alpha}\) and \(1 < s \leq t < \infty\). If
\[
\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{d} \quad \text{and} \quad \frac{p}{q} = \frac{s}{t},
\]
then there exists a constant \(C > 0\) such that
\[
\|I_{\alpha}f\|_{M_q^t} \leq C\|f\|_{M_p^s},
\]
for every \(f \in M_q^s\).

As in [3], Theorem 1.1 may be proved by using Morrey norm estimates for the Hardy-Littlewood maximal operator \(M\), which is defined by
\[
Mf(x) := \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)| \, dy
\]
for any locally integrable function \(f\) on \(\mathbb{R}^d\).

Theorem 1.2. [3] Let \(1 < p \leq q < \infty\). Then, there exists a constant \(C > 0\) such that
\[
\|Mf\|_{M_q^p} \leq C\|f\|_{M_q^p}
\]
for every \(f \in M_q^p(\mathbb{R}^d)\).

We need to mention here that the inclusion \(M_q^p \subseteq M_q^1\) is used in the proof of Theorem 1.2. In fact, this is a special case of the inclusion property of Morrey spaces which is given in the following theorem.

Theorem 1.3. [15] For \(1 \leq p_1 \leq p_2 \leq q < \infty\), the following inclusion holds:
\[
L^q(\mathbb{R}^d) = M_q^q(\mathbb{R}^d) \subseteq M_q^{p_2}(\mathbb{R}^d) \subseteq M_q^{p_1}(\mathbb{R}^d).
\]

Note that for \(d \geq 2\) and \(1 \leq p < q\), the inclusion \(M_q^q(\mathbb{R}^d) \subseteq M_q^p(\mathbb{R}^d)\) is proper. To see this, just take \(f_{p,q}(x) := |x|^{-\frac{d}{q}}\). Then one may observe that \(f_{p,q} \in M_q^p(\mathbb{R}^d) \setminus M_q^q(\mathbb{R}^d)\).

In connection with Theorem 1.3, we shall prove the inclusion properties of weak Morrey spaces, generalized Morrey spaces, and generalized weak Morrey spaces. We do not only give the sufficient condition for the inclusion properties of the generalized (weak) Morrey spaces, but also the necessary condition. The inclusion of generalized Orlicz-Morrey spaces can be seen, for instance, in [14, Theorem 4.4] and [12, Remark 1].

This paper is organized as follows. In the next section, we first discuss the inclusion property of weak Morrey spaces. After that, we shall prove the inclusion properties of generalized Morrey spaces and generalized weak Morrey spaces. Throughout this paper, we denote by \(C\) a positive constant which is independent of the function \(f\) and its value may be different from line to line.

2 Weak Morrey Spaces \(wM_q^p(\mathbb{R}^d)\)

The boundedness of fractional integral operators \(I_{\alpha}\) from Morrey spaces \(M_q^p(\mathbb{R}^d)\) to \(M_q^t(\mathbb{R}^d)\) only holds for \(1 < p \leq q < \frac{d}{\alpha}\) (and suitable \(s\) and \(t\)). The same is true on Lebesgue spaces,
the function \( I_n f \) fails to be in \( L^d_\gamma (\mathbb{R}^d) \) for \( f \in L^1(\mathbb{R}^d) \). However, weaker results are available for \( p = 1 \). In general, by weakening its membership condition, one can enlarge \( L^p(\mathbb{R}^d) \) to obtain weak Lebesgue spaces \( wL^p(\mathbb{R}^d) \).

We can do the same for Morrey spaces. We begin with the following definition.

**Definition 2.1.** Let \( 1 \leq p \leq q < \infty \). The weak Morrey space \( w\mathcal{M}^p_q(\mathbb{R}^d) \) is the set of all measurable functions \( f \) for which \( \| f \|_{w\mathcal{M}^p_q} < \infty \), where

\[
\| f \|_{w\mathcal{M}^p_q} := \sup_{a \in \mathbb{R}^d, r, \gamma > 0} |B(a, r)|^{\frac{1}{q} - \frac{1}{p}} \gamma \| \chi_{\{x : |f(x)| > \gamma\}} \|_{L^p(B(a, r))}.
\]

**Remark 2.2.** Note that \( \| \cdot \|_{w\mathcal{M}^p_q} \) defines a quasi-norm in \( w\mathcal{M}^p_q(\mathbb{R}^d) \). If \( p = q \), then \( \| \cdot \|_{w\mathcal{M}^p_q} = \| \cdot \|_{wL^p} \).

For each \( p \leq q \), the weak Morrey space \( w\mathcal{M}^p_q(\mathbb{R}^d) \) contains the Morrey space \( \mathcal{M}^p_q(\mathbb{R}^d) \), as stated in the following proposition.

**Proposition 2.3.** Let \( 1 \leq p \leq q < \infty \). Then,

\[
\mathcal{M}^p_q(\mathbb{R}^d) \subseteq w\mathcal{M}^p_q(\mathbb{R}^d),
\]

with

\[
\| f \|_{w\mathcal{M}^p_q} \leq \| f \|_{\mathcal{M}^p_q}
\]

for every \( f \in \mathcal{M}^p_q(\mathbb{R}^d) \).

**Proof.** Let \( f \in \mathcal{M}^p_q(\mathbb{R}^d) \), \( a \in \mathbb{R}^d \), \( r > 0 \), and \( \gamma > 0 \). Then, for every \( x \in B(a, r) \), we have \( \gamma \chi_{\{x : |f(x)| > \gamma\}} \leq |f(x)| \). Hence

\[
|B(a, r)|^{\frac{1}{q} - \frac{1}{p}} \gamma \chi_{\{x : |f(x)| > \gamma\}} \|_{L^p(B(a, r))} \leq |B(a, r)|^{\frac{1}{q} - \frac{1}{p}} |f| \|_{L^p(B(a, r))} \leq \| f \|_{\mathcal{M}^p_q}.
\]

By taking the supremum over all \( a, r, \) and \( \gamma \), we conclude that \( f \in w\mathcal{M}^p_q(\mathbb{R}^d) \) with \( \| f \|_{w\mathcal{M}^p_q} \leq \| f \|_{\mathcal{M}^p_q} \). \( \Box \)

**Remark 2.4.** The above inclusion is proper, i.e. there are some functions in \( w\mathcal{M}^p_q(\mathbb{R}^d) \) that are not contained in \( \mathcal{M}^p_q(\mathbb{R}^d) \). One example is the Dirac \( \delta \) function. We also note that \( f(x) := |x|^{-d/q} \in w\mathcal{M}^p_q(\mathbb{R}) \setminus \mathcal{M}^p_q(\mathbb{R}) \).

The inclusion property of weak Morrey spaces is presented in the following theorem.

**Theorem 2.5.** If \( 1 \leq p_1 \leq p_2 \leq q < \infty \), then

\[
w\mathcal{M}^{p_2}_q(\mathbb{R}^d) \subseteq w\mathcal{M}^{p_1}_q(\mathbb{R}^d)
\]

with

\[
\| f \|_{w\mathcal{M}^{p_1}_q} \leq \| f \|_{w\mathcal{M}^{p_2}_q}
\]

for every \( f \in w\mathcal{M}^{p_2}_q(\mathbb{R}^d) \).
Proof. Let \( f \in wM_{q_1}^{p_2}(\mathbb{R}^d) \), \( a \in \mathbb{R}^d \), \( r > 0 \), and \( \gamma > 0 \). By using Hölder’s inequality, we get

\[
|B(a, r)|^{\frac{1}{p} - \frac{1}{q}} \| \gamma \chi_{\{x : |f(x)| > \gamma\}} \|_{L^p(B(a, r))} \\
\leq |B(a, r)|^{\frac{1}{p} - \frac{1}{q}} \left[ \left( \int_{B(a, r)} \gamma^{p_2} \chi_{\{x : |f(x)| > \gamma\}}(x) \, dx \right)^{\frac{p_1}{p_2}} |B(a, r)|^{1 - \frac{p_1}{p_2}} \right]^{\frac{1}{p}} \\
= |B(a, r)|^{\frac{1}{p} - \frac{1}{q}} \| \gamma \chi_{\{x : |f(x)| > \gamma\}} \|_{L^{p_2}(B(a, r))} \\
\leq \| f \|_{wM_{q_1}^{p_2}}.
\]

Hence \( f \in wM_{q_1}^{p_1} \) with \( \| f \|_{wM_{q_1}^{p_1}} \leq \| f \|_{wM_{q_1}^{p_2}} \). \( \square \)

Remark 2.6. One might ask whether we can have a relation between weak Morrey spaces \( wM_{q_2}^{p_2} \) and \( wM_{q_1}^{p_1} \) for distinct values of \( q_1 \) and \( q_2 \). The answer is negative, as we shall find out in a more general setting, in the next section.

3 Inclusion of Generalized Morrey Spaces

The generalized Morrey spaces \( \mathcal{M}^p_\phi(\mathbb{R}^d) \) which we are going to define here is associated with two parameters, namely \( 1 \leq p < \infty \) and a function \( \phi : (0, \infty) \to (0, \infty) \). We assume that \( \phi \) is in the class \( \mathcal{G}_p \), that is, \( \phi \) is almost decreasing \( [r \leq s \Rightarrow \phi(r) \geq C\phi(s)] \) and \( t \mapsto t^{d/p}\phi(t) \) is almost increasing \( [r \leq s \Rightarrow t^{d/p}\phi(r) \leq C\phi(s)] \). Note that \( \phi \in \mathcal{G}_p \) implies that \( \phi \) satisfies the doubling condition, that is, there exists \( C > 0 \) such that

\[
\frac{1}{C} \leq \frac{\phi(r)}{\phi(s)} \leq C
\]

for every \( r, s > 0 \) with \( \frac{1}{2} \leq \frac{s}{r} \leq 2 \). See [4, 7, 8, 9, 15, 16] for related results.

Definition 3.1. For each \( 1 \leq p < \infty \) and \( \phi \in \mathcal{G}_p \), the generalized Morrey space \( \mathcal{M}^p_\phi(\mathbb{R}^d) \) is defined as the set of all measurable functions \( f \) on \( \mathbb{R}^d \) such that

\[
\| f \|_{\mathcal{M}^p_\phi} := \sup_{a \in \mathbb{R}^d, r > 0} \frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(x)|^p \, dx \right)^{1/p} < \infty.
\]

Observe that, if \( 1 \leq p \leq q < \infty \) and \( \phi(r) := r^{-\frac{d}{q}} \), then \( \mathcal{M}^p_\phi = \mathcal{M}^p_q \) is the classical Morrey space that we already know.

The following lemma indicates that the characteristic functions of balls are contained in the generalized Morrey spaces. This fact is proved in [5, 6]. We rewrite the proof here for convenience.

Lemma 3.2. [5, 6] Let \( 1 \leq p < \infty \) and \( \phi \in \mathcal{G}_p \). Then there exists \( C > 1 \) such that

\[
\frac{1}{\phi(r_0)} \leq \| \chi_{B_0} \|_{\mathcal{M}^p_\phi} \leq \frac{C}{\phi(r_0)}, \quad (3.1)
\]

for every ball \( B_0 := B(0, r_0) \).
Proof. Let $r_0 > 0$. By the definition of $\| \cdot \|_{\mathcal{M}^p_\phi}$, we have

$$
\| \chi_{B_0} \|_{\mathcal{M}^p_\phi} = \sup_{a \in \mathbb{R}^d, r > 0} \frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |\chi_{B_0}(x)|^p \, dx \right)^{1/p} 
\geq \frac{1}{\phi(r_0)} \left( \frac{|B_0 \cap B_0|}{|B_0|} \right)^{1/p} = \frac{1}{\phi(r_0)}.
$$

To prove the other inequality, we consider two cases. First, if $r \leq r_0$, then we have $\phi(r) \geq C \phi(r_0)$. Thus,

$$
\frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |\chi_{B_0}(x)|^p \, dx \right)^{1/p} \leq \frac{C}{\phi(r_0)} \left( \frac{|B(a, r) \cap B_0|}{|B(a, r)|} \right)^{1/q} \leq \frac{C}{\phi(r_0)}.
$$

Next, suppose that $r \geq r_0$. Since $r_0^d \phi(r_0) \leq C r^d \phi(r)$, we have

$$
\frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |\chi_{B_0}(x)|^p \, dx \right)^{1/p} \leq C r_{a, r}^d \frac{r_0^{-d}}{\phi(r_0)} \left( \frac{|B(a, r) \cap B_0|}{|B(a, r)|} \right)^{1/q} \leq C r_{a, r}^{-d} \frac{r_0^d}{\phi(r_0)} \left( \frac{|B_0|}{|B(a, r)|} \right)^{1/p} = \frac{C}{\phi(r_0)}.
$$

From these two cases, we can conclude that $\| \chi_{B_0} \|_{\mathcal{M}^p_\phi} \leq \frac{C}{\phi(r_0)}$. \qed

The inclusion property of generalized Morrey spaces is presented in the following theorem.

**Theorem 3.3.** Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, and $\phi_2 \in \mathcal{G}_{p_2}$. Then, the following statements are equivalent:

(a) $\phi_2 \leq C \phi_1$.

(b) $\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1}$.

(c) For every $f \in \mathcal{M}^{p_2}_{\phi_2}$, we have

$$
\| f \|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C \| f \|_{\mathcal{M}^{p_2}_{\phi_2}}.
$$

**Proof.** The proof of (a) implies (b) can be found in [17], and it goes as follows. Supposing that (a) holds, let $f \in \mathcal{M}^{p_2}_{\phi_2}$. For every $a \in \mathbb{R}^d$, $r > 0$, we have

$$
\frac{1}{\phi_1(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(x)|^{p_1} \, dx \right)^{1/p_1} \leq \frac{C}{\phi_2(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} (|f(x)|^{p_1})^{p_2/p_1} \, dx \right)^{p_1/p_2} \left( \int_{B(a, r)} \, dx \right)^{1 - p_1/p_2} \frac{1}{p_2} \leq \frac{C}{\phi_2(r)} \left( \int_{B(a, r)} |f(x)|^{p_2} \, dx \right)^{1/p_2} \leq C \| f \|_{\mathcal{M}^{p_2}_{\phi_2}}.
$$
Hence \( f \in \mathcal{M}^{p_1}_{\phi_1}(\mathbb{R}^d) \) with \( \|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\|f\|_{\mathcal{M}^{p_2}_{\phi_2}} \).

Now, we shall prove that (b) implies (c) by invoking the Closed Graph Theorem. Define \( T : \mathcal{M}^{p_2}_{\phi_2} \to \mathcal{M}^{p_1}_{\phi_1} \) by \( Tf = f \). Let \( \{f_n\}_{n=1}^{\infty} \subseteq \mathcal{M}^{p_2}_{\phi_2} \) converge to \( f_0 \) in \( \mathcal{M}^{p_2}_{\phi_2} \) and \( \{Tf_n\}_{n=1}^{\infty} = \{f_n\}_{n=1}^{\infty} \) converge to \( g_0 \) in \( \mathcal{M}^{p_1}_{\phi_1} \). We shall show that \( f_0 = Tf_0 = g_0 \). Note that \( f_n \) converges to \( f_0 \) in measure in a fixed ball \( B(a, r) \). In fact, for any \( \varepsilon > 0 \), we have

\[
|\{x \in B(a, r) : |f_n(x) - f_0(x)| > \varepsilon\}| \leq \frac{1}{\varepsilon^{p_2}} \int_{B(a, r)} |f_n(x) - f_0(x)|^{p_2} \, dx \leq \frac{\phi_2(r)^{p_2} |B(a, r)|}{\varepsilon^{p_2}} \|f_n - f_0\|_{\mathcal{M}^{p_2}_{\phi_2}} \rightarrow 0
\]

as \( n \to \infty \). Consequently, there exists a subsequence \( \{f_{n_j}\}_{j=1}^{\infty} \) such that

\[
\lim_{j \to \infty} f_{n_j}(x) = f_0(x) = Tf_0(x),
\]

for almost every \( x \in B(a, r) \). By using Fatou’s lemma, we have

\[
\frac{1}{\phi_1(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |Tf_0(x) - g_0(x)|^{p_1} \, dx \right)^{1/p_1} \leq \frac{1}{\phi_1(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f_{n_j}(x) - g_0(x)|^{p_1} \, dx \right)^{1/p_1} \leq \liminf_{j \to \infty} \frac{1}{\phi_1(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f_{n_j}(x) - g_0(x)|^{p_1} \, dx \right)^{1/p_1} \leq \liminf_{j \to \infty} \|f_{n_j} - g_0\|_{\mathcal{M}^{p_1}_{\phi_1}} = 0.
\]

Hence,

\[
\|Tf_0 - g_0\|_{\mathcal{M}^{p_1}_{\phi_1}} = 0.
\]

Therefore, \( Tf_0 = g_0 \). By virtue of the Closed Graph Theorem, we conclude that \( T \) is continuous, and so there exists \( C > 0 \) such that

\[
\|Tf\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\|f\|_{\mathcal{M}^{p_2}_{\phi_2}},
\]

for every \( f \in \mathcal{M}^{p_2}_{\phi_2} \). Thus, \( \|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\|f\|_{\mathcal{M}^{p_2}_{\phi_2}} \), for every \( f \in \mathcal{M}^{p_2}_{\phi_2} \).

Finally, suppose that (c) holds. Let \( B_0 := B(0, r_0) \), where \( r_0 > 0 \). Then

\[
\|\chi_{B_0}\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\|\chi_{B_0}\|_{\mathcal{M}^{p_2}_{\phi_2}}. \tag{3.2}
\]

By using Lemma 3.2, we obtain

\[
\frac{1}{\phi_1(r_0)} \leq \|\chi_{B_0}\|_{\mathcal{M}^{p_1}_{\phi_1}}, \tag{3.3}
\]

and

\[
\|\chi_{B_0}\|_{\mathcal{M}^{p_2}_{\phi_2}} \leq \frac{C}{\phi_2(r_0)}. \tag{3.4}
\]

The inequalities (3.2), (3.3), and (3.4) imply that \( \phi_2(r_0) \leq C\phi_1(r_0) \). Since \( r_0 \) is an arbitrary positive real number, we obtain \( \phi_2 \leq C\phi_1 \) as desired. \( \square \)
Remark 3.4. (1) As a consequence of Theorem 3.3, we see that we cannot have $M_{q_2} \subseteq M_{q_1}$ for distinct values of $q_1$ and $q_2$ since we do not have the inequality $r^{-d/q_2} \leq C r^{-d/q_1}$ for every $r > 0$.

(2) We can also say something about the inequality for the fractional integral operator $I_\alpha$ on the classical Morrey spaces that is presented in Theorem 1.1:

$$\|I_\alpha f\|_{M_r^s} \leq C \|f\|_{M_p^q},$$

where $p < s$ and $q < t$. According to Theorem 3.3, with $\phi_1(r) := r^{-\frac{d}{q_1}}$ and $\phi_2(r) := r^{-\frac{d}{q_2}}$, there is no inclusion relation between the range $M_r^s$ and the domain $M_p^q$.

4 Inclusion of Generalized Weak Morrey Spaces

We now move on to the generalized weak Morrey spaces, which we define as follows.

Definition 4.1. Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. The generalized weak Morrey space $wM_p^\phi = wM_p^\phi(\mathbb{R}^d)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^d$ such that

$$\|f\|_{wM_p^\phi} := \sup_{a \in \mathbb{R}^d, r, \gamma > 0} \frac{\|\gamma \chi_{\{x : |f(x)| > \gamma\}}\|_{L^p(B(a,r))}}{\phi(r)|B(a,r)|^{1/p}} < \infty.$$

See [8, 11] for related works.

The relation between the generalized Morrey spaces and their weak type is given in the following proposition.

Proposition 4.2. Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. Then $M_p^\phi \subseteq wM_p^\phi$ with

$$\|f\|_{wM_p^\phi} \leq \|f\|_{M_p^\phi}$$

for every $f \in M_p^\phi$.

Proof. Let $f \in M_p^\phi, a \in \mathbb{R}^d, r > 0$, and $\gamma > 0$. We observe that

$$\|\gamma \chi_{\{x : |f(x)| > \gamma\}}\|_{L^p(B(a,r))} \leq \left(\int_{B(a,r)} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Dividing both side by $\phi(r)|B(a,r)|^{\frac{1}{p}}$, we get

$$\frac{\|\gamma \chi_{\{x : |f(x)| > \gamma\}}\|_{L^p(B(a,r))}}{\phi(r)|B(a,r)|^{\frac{1}{p}}} \leq \frac{1}{\phi(r)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p \, dx\right)^{\frac{1}{p}} \leq \|f\|_{M_p^\phi}.$$

Therefore, $f \in wM_p^\phi$ with $\|f\|_{wM_p^\phi} \leq \|f\|_{M_p^\phi}$. □

The following is an analog of Lemma 3.2.
Lemma 4.3. Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. Then there exists $C > 1$ such that

$$\frac{1}{\phi(r_0)} \leq \|\chi_{B_0}\|_{w\mathcal{M}_\phi^p} \leq \frac{C}{\phi(r_0)},$$

for every ball $B_0 := B(0, r_0)$.

**Proof.** Let $r_0 > 0$. By using Lemma 3.2 and Proposition 4.2, we get

$$\|\chi_{B_0}\|_{w\mathcal{M}_\phi^p} \leq \|\chi_{B_0}\|_{\mathcal{M}_\phi^p} \leq \frac{C}{\phi(r_0)}.$$

Next, by using the definition of $w\mathcal{M}_\phi^p$, we have

$$\|\chi_{B_0}\|_{w\mathcal{M}_\phi^p} \geq \frac{\gamma}{\phi(r_0)} \left(\frac{\{x : |\chi_{B_0}(x)| > \gamma\}}{|B_0|}\right)^{\frac{1}{p}} = \frac{\gamma}{\phi(r_0)} \left(\frac{|B_0|}{|B_0|}\right)^{\frac{1}{p}} = \frac{\gamma}{\phi(r_0)}$$

for every $p \in (0, 1)$. Therefore $\|\chi_{B_0}\|_{w\mathcal{M}_\phi^p} \geq \frac{1}{\phi(r_0)}$, and the lemma is proved.

Finally, we come to the inclusion property of generalized weak Morrey spaces.

**Theorem 4.4.** Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, and $\phi_2 \in \mathcal{G}_{p_2}$. Then, the following statements are equivalent:

(a) $\phi_2 \leq C\phi_1$.

(b) $w\mathcal{M}_{\phi_2}^{p_2} \subseteq w\mathcal{M}_{\phi_1}^{p_1}$.

(c) For every $f \in w\mathcal{M}_{\phi_2}^{p_2}$, we have

$$\|f\|_{w\mathcal{M}_{\phi_2}^{p_1}} \leq C\|f\|_{w\mathcal{M}_{\phi_2}^{p_2}}.$$

**Proof.** Suppose that (a) holds. Let $f \in w\mathcal{M}_{\phi_2}^{p_2}$, $a \in \mathbb{R}^d$, $r > 0$, and $\gamma > 0$. By using Hölder’s inequality, we get

$$\frac{\gamma}{\phi_1(r)} \|\chi_{\{x : |f(x)| > \gamma\}}\|_{L^{p_1}(B(a,r))} \leq \frac{C\gamma}{\phi_2(r)} \left(\int_{B(a,r)} \chi_{\{x : |f(x)| > \gamma\}}(x)dx\right)^{\frac{1}{p}} \left|B(a,r)\right|^{\frac{1-p_1}{p}} \left|B(a,r)\right|^{\frac{1-p_2}{p}} \leq \|f\|_{w\mathcal{M}_{\phi_2}^{p_2}}.$$

Since this holds for every $a \in \mathbb{R}^d$, $r > 0$, and $\gamma > 0$, we conclude that $f \in w\mathcal{M}_{\phi_2}^{p_1}$ with $\|f\|_{w\mathcal{M}_{\phi_2}^{p_1}} \leq \|f\|_{w\mathcal{M}_{\phi_2}^{p_2}}$.

Now, we shall show that (b) implies (c). The idea is the same as in Theorem 3.3. Let $T$ be the identity mapping from $w\mathcal{M}_{\phi_2}^{p_2}$ into $w\mathcal{M}_{\phi_1}^{p_1}$. Suppose that $\{f_n\}_{n=1}^\infty \subseteq w\mathcal{M}_{\phi_2}^{p_2}$ converges to $f_0$ in $w\mathcal{M}_{\phi_2}^{p_2}$ and $\{Tf_n\}_{n=1}^\infty$ converges to $g_0$ in $w\mathcal{M}_{\phi_1}^{p_1}$. Fix $B(a,r)$. Observe that $f_n$ converges to $f_0$ in measure in $B(a,r)$. Indeed, for any $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\frac{\gamma \{x \in B(a,r) : |f_n(x) - f_0(x)| > \gamma\}}{\phi_2(r)\left|B(a,r)\right|^\frac{1}{p_2}} \leq \frac{1}{\varepsilon^{\frac{1}{p_2}+1}}$$
for all $\gamma > 0$. By taking $\gamma = \varepsilon$, we have
\[|\{x \in B(a, r): |f_n(x) - f_0(x)| > \varepsilon\}| < \phi_2^p(r)|B(a, r)| \varepsilon\]
for $n > n_0$. Consequently, there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that
\[\lim_{j \to \infty} f_{n_j}(x) = f_0(x) = T f_0(x),\]
for almost every $x \in B(a, r)$. By Fatou’s lemma in weak Lebesgue spaces (see [2, Lemma 5.1]), we have
\[\gamma |\{x \in B(a, r): |T f_0(x) - g_0(x)| > \gamma\}|^{\frac{1}{p_1}} \leq \frac{\liminf_{j \to \infty} f_{n_j} - g_0}{\phi_1(r)|B(a, r)|^{1/p_1}} \leq \liminf_{j \to \infty} \frac{f_{n_j} - g_0}{\phi_1(r)|B(a, r)|^{1/p_1}} \leq \liminf_{j \to \infty} \|f_{n_j} - g_0\|_{wM^{p_1}_{\phi_1}} = 0.\]
Hence,
\[\|T f_0 - g_0\|_{wM^{p_1}_{\phi_1}} = 0.\]
Therefore, $T f_0 = g_0$. It follows from the Closed Graph Theorem that $T$ is continuous, and hence, there exists $C > 0$ such that
\[\|T f\|_{wM^{p_1}_{\phi_1}} \leq C \|f\|_{wM^{p_2}_{\phi_2}},\]
for every $f \in wM^{p_2}_{\phi_2}$. Thus, $\|f\|_{wM^{p_1}_{\phi_1}} \leq C \|f\|_{wM^{p_2}_{\phi_2}}$ for every $f \in wM^{p_2}_{\phi_2}$.

Finally, suppose that (c) holds, and let $r_0 > 0$. Then, for $B_0 := B(0, r_0)$, we have
\[\|\chi_{B_0}\|_{wM^{p_1}_{\phi_1}} \leq C \|\chi_{B_0}\|_{wM^{p_2}_{\phi_2}}.\]  
(4.1)
By using Lemma 4.3, we get
\[\frac{1}{\phi_1(r_0)} \leq \|\chi_{B_0}\|_{wM^{p_1}_{\phi_1}}\]  
(4.2)
and
\[\|\chi_{B_0}\|_{wM^{p_2}_{\phi_2}} \leq \frac{C}{\phi_2(r_0)}.\]  
(4.3)
It now follows from the inequalities (4.1), (4.2), and (4.3) that $\phi_2(r_0) \leq C \phi_1(r_0)$. Since $r_0 > 0$ is arbitrary, we conclude that $\phi_2 \leq C \phi_1$. \hfill \qedsymbol

Remark 4.5. It follows from Theorem 4.4 that there isn’t an inclusion relation between $wM^{p_2}_{q_2}$ and $wM^{p_1}_{q_1}$ for distinct values of $q_1$ and $q_2$.

5 Completing the Chain of Inclusions

When the parameters are the same, we know that the generalized Morrey spaces are contained in generalized weak Morrey spaces. Together with Theorems 3.3 and 4.4, we have the following inclusion relations
\[
\begin{align*}
&M^{p_2}_{\phi_2} \quad \downarrow \quad M^{p_1}_{\phi_1} \\
&wM^{p_2}_{\phi_2} \quad \downarrow \quad wM^{p_1}_{\phi_1}
\end{align*}
\]
for $1 \leq p_1 \leq p_2 < \infty$ and $\phi_2 \leq C\phi_1$, where the arrows mean ‘contained in’. One question remains: what is the relation between $wM_{p_2}^{p_3}$ and $M_{\phi_1}^{p_1}$ for $1 \leq p_1 \leq p_2 < \infty$ and $\phi_2 \leq C\phi_1$? The answer is given in following theorem.

**Theorem 5.1.** Let $1 \leq p_1 < p_2 < \infty$, $\phi_1 \in G_{p_1}$, and $\phi_2 \in G_{p_2}$. If $\phi_2 \leq C\phi_1$, then $wM_{\phi_2}^{p_2} \subseteq M_{\phi_1}^{p_1}$ with

$$
\|f\|_{M_{p_1}^{\phi_1}} \leq C \left( \frac{p_1}{p_2 - p_1} \right)^{1/p_2} \|f\|_{wM_{\phi_2}^{p_2}}
$$

(5.1)

for all $f \in wM_{\phi_2}^{p_2}$. Conversely, if $wM_{\phi_2}^{p_2} \subseteq M_{\phi_1}^{p_1}$, then it is necessary that $\phi_2 \leq C\phi_1$.

**Proof.** The second part of the theorem follows directly from the hypothesis and Theorems 3.3 or 4.4. To prove the first part, we use the idea from [10, Exercise 1.1.11]. Let $f \in wM_{\phi_2}^{p_2}$ and $B = B(a, r)$ be any ball in $\mathbb{R}^d$. By the distribution formula, the definition of $\| \cdot \|_{wM_{\phi_2}^{p_2}}$, and $\phi_2 \leq C\phi_1$, we get

$$
\int_{B(a, r)} |f(y)|^{p_1} \, dy = p_1 \int_0^\infty \gamma^{p_1 - 1} \{x \in B : |f(x)| > \gamma\} \, d\gamma
$$

$$
= p_1 \int_0^R \gamma^{p_1 - 1} \{x \in B : |f(x)| > \gamma\} \, d\gamma + p_1 \int_R^\infty \gamma^{p_1 - 1} \{x \in B : |f(x)| > \gamma\} \, d\gamma
$$

$$
\leq p_1 |B(a, r)| \int_0^R \gamma^{p_1 - 1} \, d\gamma + p_1 \phi_2(r)^{p_2} |B(a, r)| \|f\|_{wM_{\phi_2}^{p_2}}^{p_1 - p_2} \int_R^\infty \gamma^{p_1 - p_2 - 1} \, d\gamma
$$

Therefore,

$$
\frac{1}{|B(a, r)|} \int_{B(a, r)} |f(y)|^{p_1} \, dy \leq R^p + \frac{p_1}{p_2 - p_1} \phi_1(r)^{p_2} \|f\|_{wM_{\phi_2}^{p_2}}^{p_2 - p_1}. 
$$

(5.2)

The quantity

$$
R = \left( \frac{p_1}{p_2 - p_1} \right)^{1/p_2} \phi_1(r)^{p_2} \|f\|_{wM_{\phi_2}^{p_2}}
$$

will minimize the right-hand side of the inequality (5.2). This choice of $R$ yields

$$
\frac{1}{|B(a, r)|} \int_{B(a, r)} |f(y)|^{p_1} \, dy \leq 2 \left( \frac{p_1}{p_2 - p_1} \right)^{p_1/p_2} \phi_1(r)^{p_1} \|f\|_{wM_{\phi_2}^{p_2}}^{p_1/p_2}.
$$

Hence,

$$
\frac{1}{\phi_1(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(y)|^{p_1} \, dy \right)^{1/p_1} \leq 2^{1/p_1} \left( \frac{p_1}{p_2 - p_1} \right)^{1/p_2} \|f\|_{wM_{\phi_2}^{p_2}}.
$$

By taking supremum over all balls $B$, we obtain the inequality (5.1).

**Remark 5.2.** In total, if we assume $1 \leq p_1 < p_2 < \infty$ and $\phi_2 \leq C\phi_1$, then we get the following chain of inclusions:

$$
M_{\phi_2}^{p_2} \subseteq wM_{\phi_2}^{p_2} \subseteq M_{\phi_1}^{p_1} \subseteq wM_{\phi_1}^{p_1}.
$$
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References


