DISCRETE MORREY SPACES AND THEIR GENERALIZATIONS

HENDRA GUNAWAN¹ AND EDER KIKIANTY²

Abstract. We discuss discrete Morrey spaces and their generalizations, and prove the inclusion property among these spaces through an estimate for the characteristic sequences.

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1. Introduction: Discrete Morrey Spaces

Many operators that are initially studied on Lebesgue spaces $L^p(\mathbb{R}^d)$ have discrete analogues on $\ell^p(\mathbb{Z}^d)$, see for examples [4, 5, 7, 10, 11, 12, 13]. Some of these operators have also been studied on ‘continuous’ Morrey spaces $M^p_q(\mathbb{R}^d)$, see for examples [1, 2, 3, 6, 9]. In this paper, we are interested in studying discrete analogues of Morrey spaces and their generalizations. In particular, we discuss the inclusion property of these spaces and prove some necessary and sufficient conditions for this property. For a related work on the continuous version, see [8].

To begin with, let $N \in \mathbb{N}$ and denote by $S_N$ the set $\{-N, \ldots, 0, \ldots, N\}$. Write $|S_N| = 2N + 1$ for the cardinality of $S_N$. Let $1 \leq p \leq q < \infty$. We denote by $\ell^p_q = \ell^p_q(\mathbb{Z})$ the set of real (or complex) sequences $x = (x_k)_{k \in \mathbb{Z}}$ such that

$$\|x\|_{\ell^p_q} := \sup_{N} |S_N|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_N} |x_k|^p \right)^{\frac{1}{p}} < \infty.$$  

Clearly $\ell^p_q$ is a vector space, which we shall call the discrete Morrey space. We remark that when $p = q$, we have $\ell^p_p = \ell^p$, the space of $p$-summable sequences with integer indices. For a sequence $x$ to be in $\ell^p_q$, $x$ has to have some decay, but not as fast as those in $\ell^p$. In general, for $p < q$, $\ell^p_q$ is a larger space than $\ell^p$, as stated in the following proposition.

Proposition 1. For $1 \leq p \leq q < \infty$, we have $\ell^p \subseteq \ell^p_q$ and $\|x\|_{\ell^p_q} \leq \|x\|_{\ell^p}$ for every $x \in \ell^p$.

Proof. We have for all $N \in \mathbb{N}$, $0 < |S_N|^{\frac{1}{q} - \frac{1}{p}} \leq 1$, and thus

$$|S_N|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_N} |x_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k \in S_N} |x_k|^p \right)^{\frac{1}{p}}.$$  

Taking the supremum over $N \in \mathbb{N}$, we get $\|x\|_{\ell^p_q} \leq \|x\|_{\ell^p}$.

Through the example below, we show that the above inclusion is strict.

Example 2. Let $1 \leq p < q < \infty$. Consider the sequence $x = (x_k)_{k \in \mathbb{Z}}$ given by $x_k = |k|^{-1/q}$ when $k \neq 0$ and $x_0 = 0$. Since $p/q < 1$, the series

$$\sum_{k \in \mathbb{Z}} |x_k|^p = 2 \sum_{k=1}^{\infty} \frac{1}{k^{p/q}}$$

is not absolutely convergent, and hence $x \notin \ell^p_p$. However, $x \in \ell^p_q$ since

$$\|x\|_{\ell^p_q} = \sup_{N} |S_N|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_N} |x_k|^p \right)^{\frac{1}{p}} < \infty.$$  

For a related work on the continuous version, see [8].
is divergent, thus \( x \not\in \ell^p(\mathbb{Z}) \). Next, for all \( N \in \mathbb{N} \), we have
\[
\sum_{k \in S_N} |x_k|^p = \sum_{k \in S_N, k \neq 0} \frac{1}{|k|^\frac{p}{q}} = 2 \sum_{k=1}^N \frac{1}{k^\frac{p}{q}}.
\]
Using the lower Riemann sum of \( \int_1^N x^{-\frac{q}{p}} \, dx \), we conclude that
\[
\sum_{k=1}^N \frac{1}{k^\frac{p}{q}} = 1 + \sum_{k=2}^N \frac{1}{k^\frac{p}{q}} \leq 1 + \int_1^N \frac{1}{x^\frac{q}{p}} \, dx = 1 - \frac{q}{q-p} + \frac{q}{q-p}N^{1-\frac{q}{p}}.
\]
Therefore,
\[
|S_N|^\frac{q}{p} - 1 \sum_{k \in S_N} |x_k|^p \leq 2 \left( 1 - \frac{q}{q-p} + \frac{q}{q-p}N^{1-\frac{q}{p}} \right).
\]
Since
\[
2 \left( 1 - \frac{q}{q-p} + \frac{q}{q-p}N^{1-\frac{q}{p}} \right) \rightarrow \frac{q}{q-p} - 2\frac{q}{p}, \quad \text{as } N \rightarrow \infty,
\]
and the fact that the mapping
\[
N \mapsto 2 \left( 1 - \frac{q}{q-p} + \frac{q}{q-p}N^{1-\frac{q}{p}} \right)
\]
is increasing on \( \mathbb{N} \), we have
\[
\sup_N |S_N|^\frac{q}{p} \left( \sum_{k \in S_N} |x_k|^p \right)^\frac{1}{\frac{q}{p}} \leq 2 \left( \frac{q}{q-p} \right)^\frac{1}{\frac{q}{p}},
\]
and thus \( x \in \ell^q \).

**Proposition 3.** For \( 1 \leq p \leq q < \infty \), the mapping \( \| \cdot \|_{\ell^q} \) defines a norm on \( \ell^q \). Moreover, \( (\ell^p, \| \cdot \|_{\ell^q}) \) is a Banach space.

**Proof.** It is easy to see that \( \| x \|_{\ell^q} \geq 0 \) for all \( x \in \ell^q \), and that \( x = 0 \) implies that \( \| x \|_{\ell^q} = 0 \). Let \( \| x \|_{\ell^q} = 0 \), that is,
\[
\sup_N |S_N|^\frac{q}{p} \left( \sum_{k \in S_N} |x_k|^p \right)^\frac{1}{\frac{q}{p}} = 0.
\]
Since \( |S_N|^\frac{q}{p} \left( \sum_{k \in S_N} |x_k|^p \right)^\frac{1}{\frac{q}{p}} \geq 0 \) for all \( N \in \mathbb{N} \) and \( k \in S_N \), we obtain \( x_k = 0 \) for all \( N \in \mathbb{N} \) and \( k \in S_N \), and thus \( x = 0 \). Next, let \( x \in \ell^q \) and \( \alpha \in \mathbb{R} \). Then,
\[
\| \alpha x \|_{\ell^q} = \sup_N |S_N|^\frac{q}{p} \left( \sum_{k \in S_N} |\alpha x_k|^p \right)^\frac{1}{\frac{q}{p}} = |\alpha| \| x \|_{\ell^q}.
\]
Now, let \( x, y \in \ell^q \) and \( N \in \mathbb{N} \). By using Minkowski’s inequality, we have
\[
|S_N|^\frac{q}{p} \left( \sum_{k \in S_N} |x_k + y_k|^p \right)^\frac{1}{\frac{q}{p}} \leq |S_N|^\frac{q}{p} \left( \sum_{k \in S_N} |x_k|^p \right)^\frac{1}{\frac{q}{p}} + |S_N|^\frac{q}{p} \left( \sum_{k \in S_N} |y_k|^p \right)^\frac{1}{\frac{q}{p}}.
\]
Taking the supremum over $N \in \mathbb{N}$, we get $\|x + y\|_{p_0} \leq \|x\|_{p_0} + \|y\|_{p_0}$. All these show that $\| \cdot \|_{p_0}$ defines a norm on $\ell^p_\text{q}.$

We shall now show that $(\ell^p_\text{q}, \| \cdot \|_{p_0})$ is a Banach space. Let $\varepsilon > 0$ and $(x^{(m)})_{m=1}^\infty$ be a Cauchy sequence in $\ell^p_\text{q}$. Then, there exists $M_\varepsilon \in \mathbb{N}$ such that

\begin{equation}
\sup_N |S_N|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_N} |x_k^{(m)} - x_k^{(n)}|^p \right)^{\frac{1}{p}} < \varepsilon, \quad m, n \geq M_\varepsilon.
\end{equation}

For all $k \in S_N$ and $N \in \mathbb{N},$

\begin{equation}
|S_N|^{\frac{1}{q} - \frac{1}{p}} |x_k^{(m)} - x_k^{(n)}| < \varepsilon, \quad m, n \geq M_\varepsilon.
\end{equation}

Thus, $(x_k^{(m)})_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$) for all $k \in \mathbb{Z}$. Define $x = (x_k)_{k \in \mathbb{Z}}$ such that

$x_k = \lim_{m \to \infty} x_k^{(m)}, \quad \text{for all } k \in \mathbb{Z}.$

If we let $n \to \infty$ in (1), then for all $m \geq M_\varepsilon$, we have

\begin{equation}
\sup_N |S_N|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_N} |x_k^{(m)} - x_k|^p \right)^{\frac{1}{p}} < \varepsilon.
\end{equation}

Hence, $x = x^{(m)} - (x^{(m)} - x)$ is in $\ell^p_\text{q}$ and the above result shows that $x^{(m)}$ converges to $x$ as $m \to \infty$. □

To study the relation between two discrete Morrey spaces, the following lemma will be useful.

**Lemma 4.** For all $1 \leq p_1 \leq p_2 < \infty$ and $N \in \mathbb{N}$, we have

\begin{equation}
\left( \frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^{p_1} \right)^{\frac{1}{p_1}} \leq \left( \frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^{p_2} \right)^{\frac{1}{p_2}},
\end{equation}

where $x_k, y_k \in \mathbb{R}$ for all $k \in S_N$.

**Proof.** By Hölder’s inequality, we have

\begin{equation}
\sum_{k \in S_N} |x_k|^{p_1} \leq \left( \sum_{k \in S_N} |x_k|^{p_2} \right)^{\frac{p_1}{p_2}} \left( \sum_{k \in S_N} 1 \right)^{1 - \frac{p_1}{p_2}}
= |S_N|^{1 - \frac{p_1}{p_2}} \left( \sum_{k \in S_N} |x_k|^{p_2} \right)^{\frac{p_1}{p_2}} = |S_N| \left( \frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^{p_2} \right)^{\frac{p_1}{p_2}}.
\end{equation}

Thus,

\begin{equation}
\frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^{p_1} \leq \left( \frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^{p_2} \right)^{\frac{p_1}{p_2}},
\end{equation}

and this completes the proof. □

**Proposition 5.** For all $1 \leq p_1 \leq p_2 \leq q < \infty$, we have $\ell^p_\text{q} \subseteq \ell^p_\text{q}$, and $\|x\|_{p_0} \leq \|x\|_{p_0}$ for every $x \in \ell^p_\text{q}$.

**Proof.** The proof follows immediately from Lemma 4. □
2. Weak Type Discrete Morrey spaces

For $1 \leq p \leq q < \infty$, we define the weak type discrete Morrey space $w\ell_p^q$ to be the set of all real (or complex) sequences $x = (x_k)_{k \in \mathbb{Z}}$ for which $\|x\|_{w\ell_p^q} < \infty$, where

$$
\|x\|_{w\ell_p^q} = \sup_{N \in \mathbb{N}, \gamma > 0} |S_N|^{rac{1}{q} - \frac{1}{p}} \gamma \left| \{ k \in S_N : |x_k| > \gamma \} \right|^{\frac{1}{p}}.
$$

Note that when $p = q$, we have $\ell_p^p := w\ell_p^p$, which is the weak type $\ell^p$ space. The following example shows that the weak type spaces have more elements.

**Example 6.** The sequence $x = (x_k)_{k \in \mathbb{Z}}$ given by $x_k = |k|^{-1/p}$ when $k \neq 0$ and $x_0 = 0$ is not in $\ell^p$. Nevertheless, for any $\gamma > 0$, we have

$$
\gamma \left| \{ k \in S_N : |k|^{-1} > \gamma \} \right|^{\frac{1}{p}} = 2\gamma \left| \{ k \in \mathbb{N} : 1 \leq k \leq N, |k|^{-1/p} > \gamma \} \right|^{\frac{1}{p}} < 2\gamma \cdot \frac{1}{\gamma} = 2.
$$

Thus $(x_k)_{k \in \mathbb{Z}}$ is in $w\ell_p^p$.

**Theorem 7.** For $1 \leq p \leq q < \infty$, $\ell_q^p \subset w\ell_q^p$ and $\|x\|_{w\ell_q^p} \leq \|x\|_{\ell_q^p}$ for every $x \in \ell_q^p$.

**Proof.** Let $x \in \ell_q^p$, $\gamma > 0$ and $N \in \mathbb{N}$. We have,

$$
|S_N|^{rac{1}{q} - \frac{1}{p}} \gamma \left| \{ k \in S_N : |x_k| > \gamma \} \right|^{\frac{1}{p}} = |S_N|^{rac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_N, |x_k| > \gamma} \gamma^p \right)^{\frac{1}{p}} \\
\leq |S_N|^{rac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_N, |x_k| > \gamma} |x_k|^p \right)^{\frac{1}{p}} \\
\leq |S_N|^{rac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_N} |x_k|^p \right)^{\frac{1}{p}}.
$$

Taking the supremum over $N \in \mathbb{N}$ and $\gamma > 0$, we obtain $\|x\|_{w\ell_q^p} \leq \|x\|_{\ell_q^p}$. Therefore, if $x \in \ell_q^p$, then $x \in w\ell_q^p$. $\square$

**Theorem 8.** For $1 \leq p \leq q < \infty$, $\| \cdot \|_{w\ell_q^p}$ is a quasi-norm, so that $(w\ell_q^p, \| \cdot \|_{w\ell_q^p})$ is a quasi-normed space.

**Proof.** From the definition of $\| \cdot \|_{w\ell_q^p}$, it is clear that $\|x\|_{w\ell_q^p} \geq 0$ for all $x \in w\ell_q^p$. Let $\gamma > 0$ and $N \in \mathbb{N}$. If $x = 0$, then $\{ k \in S_N : |x_k| > \gamma \}$ is an empty set and therefore its cardinality is zero, and thus

$$
|S_N|^{rac{1}{q} - \frac{1}{p}} (\gamma^p)^{\frac{1}{p}} \left| \{ k \in S_N : |x_k| > \gamma \} \right|^{\frac{1}{p}} = 0.
$$

Taking the supremum over $\gamma > 0$ and $N \in \mathbb{N}$, we obtain $\|x\|_{w\ell_q^p} = 0$. Now let $\|x\|_{w\ell_q^p} = 0$. Then,

$$
\left| \{ k \in S_N : |x_k| > \gamma \} \right| = 0, \quad \text{for all } N \in \mathbb{N} \text{ and } \gamma > 0.
$$

We conclude that for all $k \in \mathbb{Z}$, we have

$$
0 \leq |x_k| \leq \gamma, \quad \text{for all } \gamma > 0.
$$

Thus, $x_k = 0$ for all $k \in \mathbb{Z}$, that is, $x = 0$. 

Next, let $x \in w\ell^p_q$ and $\alpha \in \mathbb{R}$. When $\alpha = 0$, $\|\alpha x\|_{w\ell^p_q} = |\alpha|\|x\|_{w\ell^p_q}$. Suppose $\alpha \neq 0$. We have,

$$\|\alpha x\|_{w\ell^p_q} = \sup_{N \in \mathbb{N}, \gamma > 0} |S_N|^\frac{1}{p} - \frac{1}{p} \gamma \{k \in S_N: |\alpha x_k| > \gamma\}^\frac{1}{p}$$

$$= \sup_{N \in \mathbb{N}, \gamma > 0} |S_N|^\frac{1}{p} - \frac{1}{p} \gamma \left\{k \in S_N: |x_k| > \frac{\gamma}{|\alpha|}\right\}^\frac{1}{p}$$

$$= \sup_{N \in \mathbb{N}, \gamma > 0} |S_N|^\frac{1}{p} - \frac{1}{p} \gamma \|x\|_{w\ell^p_q}$$

Now, let $x, y \in w\ell^p_q$. For any $N \in \mathbb{N}$, we consider $k \in S_N$ such that $|x_k| \leq |y_k|$, and we have the inequality

$$|x_k + y_k| \leq |x_k| + |y_k| \leq 2|y_k|.$$  

Similarly, for $k \in S_N$ such that $|x_k| > |y_k|$, we have the inequality

$$|x_k + y_k| \leq |x_k| + |y_k| \leq 2|x_k|.$$  

Therefore, we have the inclusions

$$\{k \in S_N: |x_k + y_k| > \gamma\} \subseteq \{k \in S_N: |x_k| + |y_k| > \gamma\} \subseteq \{k \in S_N: 2|x_k| > \gamma\} \cup \{k \in S_N: 2|y_k| > \gamma\}.$$  

For any $\gamma > 0$ and $N \in \mathbb{N}$, we have

$$\left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \right)^p \{k \in S_N: |x_k + y_k| > \gamma\}$$

$$\leq \left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \right)^p \{k \in S_N: 2|x_k| > \gamma\} + \left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \right)^p \{k \in S_N: 2|y_k| > \gamma\}.$$  

By writing $\delta = \gamma/2$ on the right hand side of the above inequality, we get

$$\left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \right)^p \{k \in S_N: |x_k + y_k| > \gamma\}$$

$$\leq 2^p \left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \right)^p \{k \in S_N: |x_k| > \gamma\} + 2^p \left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \right)^p \{k \in S_N: |y_k| > \gamma\},$$

or equivalently,

$$|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \{k \in S_N: |x_k + y_k| > \gamma\}^\frac{1}{p}$$

$$\leq 2 \left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \right)^p \{k \in S_N: |x_k| > \gamma\} + \left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \right)^p \{k \in S_N: |y_k| > \gamma\}^\frac{1}{p}$$

$$\leq 2 \left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \delta \right)^p \{k \in S_N: |x_k| > \gamma\} + 2 \left(|S_N|^\frac{1}{p} - \frac{1}{p} \gamma \delta \right)^p \{k \in S_N: |y_k| > \gamma\}.$$  

Taking the supremum over $\gamma > 0$ and $N \in \mathbb{N}$, we get $\|x + y\|_{w\ell^p_q} \leq 2(\|x\|_{w\ell^p_q} + \|y\|_{w\ell^p_q})$.

**Remark.** At this stage we do not know whether $w\ell^p_q$ is complete or not with respect to the quasi-norm $\|\cdot\|_{w\ell^p_q}$.

The following proposition gives the inclusion property between two weak type discrete Morrey spaces.

**Proposition 9.** Let $1 \leq p_1 \leq p_2 < q < \infty$. Then, $w\ell^p_{q1} \subseteq w\ell^p_{q2}$ and $\|x\|_{w\ell^p_{q1}} \leq \|x\|_{w\ell^p_{q2}}$ for every $x \in w\ell^p_{q2}$.

**Proof.** Let $x \in w\ell^p_{q1}$ and $\gamma > 0$. By definition, we have

$$|S_N|^\frac{1}{p} - \frac{1}{p} \gamma |\{k \in S_N: |x_k| > \gamma\}|^\frac{1}{p} \leq \|x\|_{w\ell^p_{q1}}.$$
for any $N \in \mathbb{N}$. Equivalently, we have

$$
\gamma \leq \frac{|S_N|^{\frac{1}{p} - \frac{1}{q}}}{\left|\left\{ k \in S_N : |x_k| > \gamma \right\}\right|^{\frac{1}{p}} \| w \|_{\ell_q^q}^q.
$$

Therefore, for any $N \in \mathbb{N}$, we have

$$
|S_N|^{\frac{1}{q} - \frac{1}{p}} \left|\left\{ k \in S_N : |x_k| > \gamma \right\}\right|^{\frac{1}{q}} \leq \frac{|S_N|^{\frac{1}{p} - \frac{1}{q}}}{\left|\left\{ k \in S_N : |x_k| > \gamma \right\}\right|^{\frac{1}{p}} \| w \|_{\ell_q^q}^q
$$

$$
= \left( \frac{\left|\left\{ k \in S_N : |x_k| > \gamma \right\}\right|}{|S_N|} \right)^{\frac{1}{p} - \frac{1}{q}} \| w \|_{\ell_q^q}^q
$$

\leq \| w \|_{\ell_q^q}^q.

Taking the supremum over $N \in \mathbb{N}$ and $\gamma > 0$, we obtain $\| w \|_{\ell_q^q}^q \leq \| w \|_{\ell_q^q}^q$, and the proof is complete. \qed

3. Generalized Discrete Morrey Spaces

By $\leq$ and $\geq$ we mean that the inequalities are satisfied up to a constant $C > 0$, that is, $x \leq y$ means $x \leq Cy$ for some $C > 0$.

The generalized discrete Morrey space $\ell_p^p$ is equipped with two parameters, that is, $1 \leq p < \infty$ and a function $\phi \in \mathcal{G}_p$, where $\mathcal{G}_p$ is the set of all function $\phi : (0, \infty) \to (0, \infty)$ such that $\phi$ is almost decreasing ($r \leq s$ implies that $\phi(r) \gtrsim \phi(s)$), and the mapping $t \mapsto t^\phi \phi(t)$ is almost increasing (that is, $r \leq s$ implies that $r^\phi \phi(r) \lesssim s^\phi \phi(s)$). Note that $\phi \in \mathcal{G}_p$ implies that $\phi$ satisfies the doubling condition, that is, there exists $C > 0$ such that

$$
\frac{1}{C} \leq \frac{\phi(r)}{\phi(s)} \leq C
$$

for every $\frac{1}{2} \leq \frac{r}{s} \leq 2$.

For $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$, the generalized discrete Morrey space $\ell_p^p$ is defined as the set of all real (or complex) sequences $x = (x_k)_{k=1}^\infty$ such that

$$
\| x \|_{\ell_p^p} = \sup_N \frac{1}{\phi(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^p \right)^{\frac{1}{p}} < \infty.
$$

Note that the discrete Morrey space $\ell_p^p$ ($1 \leq p \leq \infty$) may be obtained from $\ell_p^p$ by choosing the function $\phi(N) = |S_N|^{\frac{1}{p}}$.

**Lemma 10.** Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. Let $N_0 \in \mathbb{N}$, and $\xi^{N_0}$ be a sequence for which

$$
\xi^{N_0}_k = \begin{cases} 
1, & \text{if } k \in S_{N_0}; \\
0, & \text{otherwise}.
\end{cases}
$$

Then,

$$
\frac{1}{\phi(N_0)} \leq \| \xi^{N_0} \|_{\ell_p^p} \leq \frac{1}{\phi(N_0)}.
$$

**Proof.** We have

$$
\| \xi^{N_0} \|_{\ell_p^p} = \sup_N \frac{1}{\phi(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N} |\xi^{N_0}_k|^p \right)^{\frac{1}{p}} \geq \frac{1}{\phi(N_0)} \left( \frac{|S_{N_0}|}{|S_{N_0}|} \right)^{\frac{1}{p}} = \frac{1}{\phi(N_0)}.
$$

For the second inequality, we split the proof into two cases:
Case 1: When \( N \leq N_0 \), we have \( \phi(N) \gtrsim \phi(N_0) \) and \( \sum_{k \in S_N} |\xi_k^{N_0}|^p = |S_N| \). Therefore,
\[
\frac{1}{\phi(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N} |\xi_k^{N_0}|^p \right)^{\frac{1}{p}} \leq \frac{1}{\phi(N_0)} \left( \frac{|S_N|}{|S_N|} \right)^{\frac{1}{p}} \leq \frac{1}{\phi(N_0)}
\]

Case 2: When \( N \geq N_0 \), we have \( N^{\frac{1}{p}} \phi(N_0) \lesssim N^{\frac{1}{p}} \phi(N) \) and thus
\[
\frac{1}{\phi(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N} |\xi_k^{N_0}|^p \right)^{\frac{1}{p}} \lesssim \frac{N^{\frac{1}{p}} N_0^{\frac{1}{p}}}{\phi(N_0)} \left( \frac{|S_N|}{|S_N|} \right)^{\frac{1}{p}} = \frac{1}{\phi(N_0)} \left( \frac{N}{N_0} \right)^{\frac{1}{p}} \left( \frac{2N_0 + 1}{2N + 1} \right)^{\frac{1}{p}} \leq \frac{1}{\phi(N_0)} \left( \frac{N}{N_0} \right)^{\frac{1}{p}} \left( \frac{3N_0}{2N} \right)^{\frac{1}{p}} \lesssim \frac{1}{\phi(N_0)}.
\]

This completes the proof. \( \square \)

**Theorem 11.** Let \( 1 \leq p_1 \leq p_2 < \infty \), \( \phi_1 \in \mathcal{G}_{p_1} \), and \( \phi_2 \in \mathcal{G}_{p_2} \). Then, the following statements are equivalent:

(i) \( \phi_2(N) \lesssim \phi_1(N) \), for all \( N \in \mathbb{N} \).

(ii) \( \ell_{p_2}^0 \subseteq \ell_{p_1}^0 \), with \( \|x\|_{\ell_{p_1}^0} \lesssim \|x\|_{\ell_{p_2}^0} \) for every \( x \in \ell_{p_2}^0 \).

**Proof.** Suppose that (i) holds. Let \( x \in \ell_{p_2}^0 \). For any \( N \in \mathbb{N} \), we have
\[
\frac{1}{\phi_1(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^{p_1} \right)^{\frac{1}{p_1}} \lesssim \frac{1}{\phi_2(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^{p_1} \right)^{\frac{1}{p_1}} \lesssim \frac{1}{\phi_2(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N} |x_k|^{p_2} \right)^{\frac{1}{p_2}}.
\]
Note the use of Lemma 4 in the last inequality. Taking the supremum over \( N \in \mathbb{N} \), we obtain \( \|x\|_{\ell_{p_1}^0} \lesssim \|x\|_{\ell_{p_2}^0} \).

Now suppose that (ii) holds. Let \( N_0 \in \mathbb{N} \), and \( \xi_k^{N_0} \) be a sequence defined by (2) as in Lemma 10. Then, \( \|\xi_k^{N_0}\|_{\ell_{p_1}^0} \lesssim \|\xi_k^{N_0}\|_{\ell_{p_2}^0} \) by our assumption. Lemma 10 gives us
\[
\frac{1}{\phi_1(N_0)} \lesssim \|\xi_k^{N_0}\|_{\ell_{p_1}^0} \quad \text{and} \quad \|\xi_k^{N_0}\|_{\ell_{p_2}^0} \lesssim \frac{1}{\phi_2(N_0)}.
\]
We conclude that
\[
\frac{1}{\phi_1(N_0)} \lesssim \frac{1}{\phi_2(N_0)}, \quad \text{or equivalently,} \quad \phi_2(N_0) \lesssim \phi_1(N_0);
\]
and this completes the proof since we choose arbitrary \( N_0 \in \mathbb{N} \). \( \square \)

**Remark.** As in the continuous case (see [8]), Theorem 11 may be sharpened by replacing (ii) by the statement that \( \ell_{p_2}^0 \subseteq \ell_{p_1}^0 \), without information about the norms. The reason is because the inequality \( \| \cdot \|_{\ell_{p_1}^0} \lesssim \| \cdot \|_{\ell_{p_2}^0} \) holds once we know the inclusion between two Banach lattices \( \ell_{p_2}^0 \) and \( \ell_{p_1}^0 \).
4. Generalized Weak Type Discrete Morrey Spaces

For $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$, the generalized weak type discrete Morrey space $w\ell^p_\phi$ is the set of all real (or complex) sequences $x = (x_k)_{k \in \mathbb{Z}}$ for which $\|x\|_{w\ell^p_\phi} < \infty$, and $\| \cdot \|_{w\ell^p_\phi}$ is defined by

$$\|x\|_{w\ell^p_\phi} = \sup_{N \in \mathbb{N}, \gamma > 0} \frac{\gamma}{\phi(N)} \left( \frac{|\{k \in S_N : |x_k| > \gamma\}|}{|S_N|} \right)^{\frac{1}{p}}.$$

**Proposition 12.** For $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$, $\ell^p_\phi \subseteq w\ell^p_\phi$ and $\|x\|_{w\ell^p_\phi} \leq \|x\|_{\ell^p_\phi}$ for every $x \in \ell^p_\phi$.

**Proof.** Let $x \in \ell^p_\phi$ and $\gamma > 0$ and $N \in \mathbb{N}$. We have,

$$\frac{\gamma}{\phi(N)} \left( \frac{|\{k \in S_N : |x_k| > \gamma\}|}{|S_N|} \right)^{\frac{1}{p}} = \frac{1}{\phi(N)} \left( \frac{\gamma^p |\{k \in S_N : |x_k| > \gamma\}|}{|S_N|} \right)^{\frac{1}{p}}$$

$$= \frac{1}{\phi(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N : |x_k| > \gamma} \gamma^p \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{\phi(N)} \left( \frac{1}{|S_N|} \sum_{k \in S_N : |x_k| > \gamma} |x_k|^p \right)^{\frac{1}{p}}$$

$$= \frac{1}{\phi(N)} \left( \frac{1}{|S_N|} \left| \sum_{k \in S_N} |x_k|^p \right| \right)^{\frac{1}{p}}.$$

Taking the supremum over $N \in \mathbb{N}$ and $\gamma > 0$, we obtain $\|x\|_{w\ell^p_\phi} \leq \|x\|_{\ell^p_\phi}$. Therefore, if $x \in \ell^p_\phi$, then $x \in w\ell^p_\phi$. □

**Lemma 13.** Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. If $N_0 \in \mathbb{N}$, and $\xi^{N_0}$ is a sequence defined by (2) as in Lemma 10. Then,

$$\frac{1}{2\phi(N_0)} \leq \|\xi^{N_0}\|_{w\ell^p_\phi} \lesssim \frac{1}{\phi(N_0)}.$$

**Proof.** By definition,

$$\|\xi^{N_0}\|_{w\ell^p_\phi} = \sup_{N \in \mathbb{N}, \gamma > 0} \frac{\gamma}{\phi(N)} \left( \frac{|\{k \in S_N : |\xi_k^{N_0}| > \gamma\}|}{|S_N|} \right)^{\frac{1}{p}}$$

$$\geq \frac{1/2}{\phi(N_0)} \left( \frac{|\{k \in S_{N_0} : |\xi_k^{N_0}| > \frac{1}{2}\}|}{|S_{N_0}|} \right)^{\frac{1}{p}}$$

$$\geq \frac{1}{2\phi(N_0)} \left( \frac{|S_{N_0}|}{|S_{N_0}|} \right)^{\frac{1}{p}} = \frac{1}{2\phi(N_0)}.$$

By using Lemma 10 and Proposition 12, we have

$$\|\xi^{N_0}\|_{w\ell^p_\phi} \leq \|\xi^{N_0}\|_{\ell^p_\phi} \lesssim \frac{1}{\phi(N_0)}$$

and this completes the proof. □

**Theorem 14.** Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$ and $\phi_2 \in \mathcal{G}_{p_2}$. Then, the following statements are equivalent:

(i) $\phi_2(N) \lesssim \phi_1(N)$, for all $N \in \mathbb{N}$. 

(ii) \( w^{p_2}_{\phi_2} \subseteq w^{p_1}_{\phi_1} \) with \( \|x\|_{w^{p_1}_{\phi_1}} \lesssim \|x\|_{w^{p_2}_{\phi_2}} \) for every \( x \in w^{p_2}_{\phi_2} \).

Proof. Suppose that (i) holds. Let \( x \in w^{p_2}_{\phi_2} \) and \( \gamma > 0 \). By definition, we have
\[
\frac{\gamma}{\phi_2(N)} \left( \frac{|\{k \in S_N : |x_k| > \gamma\}|}{|S_N|} \right)^{\frac{1}{p_2}} \leq \|x\|_{w^{p_2}_{\phi_2}},
\]
for any \( N \in \mathbb{N} \). Equivalently, we have
\[
\frac{\gamma}{\phi_2(N)} \leq \left( \frac{|\{k \in S_N : |x_k| > \gamma\}|}{|S_N|} \right)^{-\frac{1}{p_2}} \|x\|_{w^{p_2}_{\phi_2}}.
\]
Therefore, for any \( N \in \mathbb{N} \), we have
\[
\frac{\gamma}{\phi_1(N)} \left( \frac{|\{k \in S_N : |x_k| > \gamma\}|}{|S_N|} \right)^{\frac{1}{p_1}} \leq \frac{\gamma}{\phi_2(N)} \left( \frac{|\{k \in S_N : |x_k| > \gamma\}|}{|S_N|} \right)^{\frac{1}{p_2}} \leq \left( \frac{|\{k \in S_N : |x_k| > \gamma\}|}{|S_N|} \right)^{\frac{1}{p_1} - \frac{1}{p_2}} \|x\|_{w^{p_2}_{\phi_2}} \leq \|x\|_{w^{p_2}_{\phi_2}}.
\]
Taking the supremum over \( N \in \mathbb{N} \) and \( \gamma > 0 \), we obtain \( \|x\|_{w^{p_1}_{\phi_1}} \lesssim \|x\|_{w^{p_2}_{\phi_2}} \).

Now suppose that (ii) holds. Let \( N_0 \in \mathbb{N} \) and \( \xi^{N_0} \) be a sequence defined by (2) as in Lemma 10. Then, we have \( \|\xi^{N_0}\|_{w^{p_1}_{\phi_1}} \lesssim \|\xi^{N_0}\|_{w^{p_2}_{\phi_2}} \). By Lemma 13, we have
\[
\frac{1}{2\phi_1(N_0)} \leq \|\xi^{N_0}\|_{w^{p_1}_{\phi_1}} \quad \text{and} \quad \|\xi^{N_0}\|_{w^{p_2}_{\phi_2}} \lesssim \frac{1}{\phi_2(N_0)}.
\]
We conclude that
\[
\frac{1}{\phi_1(N_0)} \lesssim \frac{1}{\phi_2(N_0)}, \quad \text{or equivalently,} \quad \phi_2(N_0) \lesssim \phi_1(N_0),
\]
and this completes the proof since we choose arbitrary \( N_0 \in \mathbb{N} \). \( \square \)

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References


1Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia, E-mail: hgunawan@math.itb.ac.id

2Department of Pure and Applied Mathematics, University of Johannesburg, Auckland Park 2006, South Africa, E-mail: ekikianty@uj.ac.za