CARISTI AND BANACH FIXED POINT THEOREM ON PARTIAL METRIC SPACE.

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Abstract. In this article are shown Caristi and Banach fixed point theorem type in the partial metric space. Both will be proven by using Ekeland’s variational principle in partial metric space which also introduced in this article.

1. Introduction

Caristi fixed point theorem was generalized by several authors. For example, Bae [1] generalized Caristi’s theorem to prove the fixed point theorem for weakly contractive set-valued mapping as well as Banach fixed point theorem in the other way.

In recent years many work on domain theory have been made in order to equip semantics domain with a notion of distance, see [2]-[3], [6]-[9]. In particular, Matthews [8] introduced the notion of a partial metric space as a part of the study of donata- tional semantic of data flow network, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

In this paper we present the Caristi and Banach fixed point theorem type in partial metric spaces. We would also introduce Ekeland variational principle on partial metric spaces and its applications to fixed point.

2. Preliminaries

First, we start with some preliminaries on partial metric spaces. For more details, we refer to reader to [8].

Definition 2.1. Let $X$ be nonempty set. The mapping $p : X \times X \rightarrow \mathbb{R}^+$ is said partial metric on $X$ if satisfies

(P1) $p(x, x) \leq p(x, y)$ for all $x, y \in X$;
(P2) $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$;
(P3) $p(x, y) = p(y, x)$ for all $x, y \in X$;
(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ for all $x, y, z \in X$.

The pairs $(X, p)$ is called a partial metric space. Note that the self-distance of any point need not be zero. A partial metric is a metric on $X$ if $p(x, x) = 0$ for any $x \in X$.

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Example 2.2. Let \( \mathbb{R} \) be a set real number and the distance function \( p : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be defined by
\[
p(x, y) = \frac{1}{3} \left( |x - y| + |x| + |y| \right), \quad \forall x, y \in \mathbb{R}.
\]
Then \( p \) is a partial metric on \( \mathbb{R} \).

Lemma 2.3. Let \((X, p)\) be a partial metric space, and the function \( d_p : X \times X \to [0, \infty) \) be defined by
\[
d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad \forall x, y \in X.
\]
Then \( d_p \) is a metric.

Proof. (i) Clear for all \( x, y \in X \) then \( d_p(x, y) \geq 0 \).
(ii) From (P2), we have
\[
x = y \iff p(x, x) = p(y, y) = p(x, y)
\]
\[
\iff d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)
\]
\[
\iff d_p(x, y) = 2p(x, y) - 2p(x, y)
\]
\[
\iff d_p(x, y) = 0.
\]
(iii) Clear for all \( x, y \in X \), \( d_p(x, y) = d_p(y, x) \)
(iv) For all \( x, y, z \in X \) and from (P4), we obtain
\[
d_p(x, z) = 2p(x, z) - p(x, x) - p(z, z)
\]
\[
\leq 2(p(x, y) + p(y, z) - p(y, y)) - p(x, x) - p(z, z)
\]
\[
= (2p(x, y) - p(x, x) - p(y, y)) + (2p(y, z) - p(y, y) - p(z, z))
\]
\[
= d_p(x, y) + d_p(y, z).
\]
\( \square \)

The Lemma 2.3, describe that metric is a special case of partial metric. Therefore the partial metric is a generalization of metric.

Definition 2.4. Let \((X, p)\) be a partial metric space, a point \( x_0 \in X \) and \( \epsilon > 0 \). The open ball for a partial metric \( p \) are sets of the form
\[
B_{\epsilon}(x_0) = \{ x \in X \mid p(x_0, x) < \epsilon \}.
\]
Since \( p(x_0, x_0) > 0 \), the open ball are sets of the form
\[
B_{\epsilon + p(x_0, x_0)}(x_0) = \{ x \in X \mid p(x_0, x) < \epsilon + p(x_0, x_0) \}.
\]
Contrary to the metric space case, some open balls may be empty. If \( \epsilon > p(x_0, x_0) \), then \( B_{\epsilon}(x_0) = B_{\epsilon - p(x_0, x_0)}(x_0) \). If \( 0 < \epsilon \leq p(x_0, x_0) \), we obtain
\[
B_{\epsilon}(x_0) = \{ x \in X \mid p(x_0, x) < \epsilon \leq p(x_0, x_0) \} = \emptyset.
\]
This mean the open ball \( B_{p(x_0, x_0)}(x_0) \) be empty set. However may be point \( x_0 \notin B_{p(x_0, x_0)}(x_0) \).

Definition 2.5. A sequence \( \langle x_n \rangle \) in a partial metric space \((X, p)\) converges to \( x_0 \in X \) if, for any \( \epsilon > 0 \) such that \( x_0 \in B_{\epsilon}(x_0) \) there exists \( N \in \mathbb{N} \) so that for any \( n \geq N, x_n \in B_{\epsilon}(x_0) \). We write \( \lim_{n \to \infty} x_n = x_0 \).

Lemma 2.6. A sequence \( \langle x_n \rangle \) in a partial metric space \((X, p)\). Then \( \langle x_n \rangle \) converges to point \( x_0 \in X \) if and only if \( \lim_{n \to \infty} p(x_n, x_0) = p(x_0, x_0) \).
Proof. By Definition 2.5, for any $\epsilon > 0$, $p(x_n, x_0) < \epsilon$ for any $n \geq N$. Since $B_\epsilon(x_0) \neq \emptyset$ of course $p(x_0, x_0) \leq \epsilon$ this implies $p(x_n, x_0) - p(x_0, x_0) < \epsilon$, for any $n \geq N$ so that $\lim_{n \to \infty} p(x_n, x_0) = p(x_0, x_0)$.

Conversely suppose that $p(x_0, x_0) = \lim_{n \to \infty} p(x_n, x_0)$. If $x_0 \in B_\epsilon(x_0)$, then there exists $N \in \mathbb{N}$ such that for any $n \geq N, p(x_n, x_0) < \epsilon$. This mean $x_n \in B_\epsilon(x_0)$ for any $n \geq N$. By Definition 2.5, $(x_n)$ converges to point $x_0 \in X$. If $x_0 \in B_\epsilon(a)$ with $a \in X$, that is $p(a, x_0) < \epsilon$, then there exists $N \in \mathbb{N}$ such that for any $n \geq N, p(x_n, x_0) - p(x_0, x_0) < \epsilon - p(x_0, a)$ so that for any $n \geq N$ we obtain

$$p(x_n, a) \leq p(x_n, x_0) + p(x_0, a) - p(x_0, x_0) < (\epsilon - p(x_0, a)) + p(x_0, a) = \epsilon.$$ 

This means for any $n \geq N, x_n \in B_\epsilon(a)$. □

Definition 2.7. A sequence $(x_n)$ in a partial metric space $(X, p)$ is called properly converges to $x \in X$ if $(x_n)$ converges to $x$ and

$$\lim_{n \to \infty} p(x_n, x_n) = p(x, x).$$

In other words, a sequence $(x_n)$ properly converges to $x \in X$ if $\lim_{n \to \infty} p(x_n, x)$ and $\lim_{n \to \infty} p(x, x_n)$ exists and

$$\lim_{n \to \infty} p(x_n, x_n) = \lim_{n \to \infty} p(x_n, x) = p(x, x).$$

Notice that every convergent sequence in a metric space converging in partial metric spaces.

Definition 2.8. A sequence $(x_n)$ in a partial metric space $(X, p)$ is called a Cauchy sequence if $\lim_{m,n \to \infty} p(x_n, x_m)$ exists and is finite.

In other words, $(x_n)$ is Cauchy if the numbers sequence $p(x_n, x_m)$ converges to some $\lambda \in \mathbb{R}$ as $n$ and $m$ approach to infinity, that is, if $\lim_{m,n \to \infty} p(x_n, x_m) = \lambda < \infty$. This means for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N, |p(x_n, x_m) - \lambda| < \epsilon$. If $(X, p)$ is a metric space then $\lambda = 0$.

Lemma 2.9. A sequence $(x_n)$ in a partial metric space $(X, p)$. If $(x_n)$ properly converges to $x$ then $(x_n)$ is Cauchy sequence.

Proof. By Definition 2.7, $p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n \to \infty} p(x, x_n)$. By (P1) and (P4) we have

$$p(x, x) = \lim_{m,n \to \infty} p(x_n, x_n) \leq \lim_{n,m \to \infty} p(x_n, x_m) \leq \lim_{n \to \infty} p(x, x) + \lim_{n \to \infty} p(x_m, x) - p(x, x) = p(x, x).$$

Hence $\lim_{n,m \to \infty} p(x_n, x_m) = p(x, x)$. This means, there exists $\lambda \in \mathbb{R}^+$ such that $\lambda = p(x, x)$ and $\lim_{n,m \to \infty} p(x_n, x_m) = \lambda$. The sequence $(x_n)$ is Cauchy proved. □

Theorem 2.10. A sequence $(x_n)$ in a partial metric space $(X, p)$ is a Cauchy, if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have

$$p(x_n, x_m) - p(x_m, x_m) < \epsilon$$

Proof. Since $(x_n)$ is Cauchy, there exists $\lambda \in \mathbb{R}^+$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ and for all $n, m \geq N$ we have

$$|p(x_n, x_m) - \lambda| < \frac{\epsilon}{2}.$$
Let \( n = m \geq N \), then \( |p(x_m, x_m) - \lambda| < \frac{\lambda}{2} \). Therefore
\[
|p(x_n, x_m) - p(x_m, x_m)| \leq |p(x_n, x_m) - \lambda| + |p(x_m, x_m) - \lambda| < \epsilon.
\]
By (P1), we obtain \( p(x_n, x_m) - p(x_m, x_m) < \epsilon \). Conversely it is obvious. \( \square \)

**Definition 2.11.** A metric partial space is complete if every Cauchy sequence properly converges.

**Definition 2.12.** Let \((X,p)\) be a metric partial space and \(A\) be a nonempty subset of \(X\). The diameter of \(A\), denoted by \(D(A)\), is given by
\[
D(A) = \sup\{p(x,y) : x, y \in A\}
\]

**Theorem 2.13.** Let \((X,p)\) be a complete metric partial space and \(F_n\) be a decreasing sequence (that is, \(F_n \supset F_{n+1}\)) of nonempty closed subsets of \(X\) such that \(D(F_n) \rightarrow 0\) as \(n \rightarrow \infty\). Then the intersection \(\cap_{n=1}^{\infty} F_n\) contains exactly one point.

**Proof.** The first, construct a sequence \(\langle x_n \rangle\) in \(X\) by selecting a point \(x_n \in F_n\) for each \(n \in \mathbb{N}\). Since \(F_n \supset F_{n+1}\) for all \(n\), we have \(x_n \in F_n \subset F_m\) for all \(n > m\).

Let \(\epsilon > 0\) be given. Since \(D(F_n) \rightarrow 0\), there exists \(N \in \mathbb{N}\) such that \(D(F_n) < \epsilon\) for each \(n \geq N\). Since \(F_m, F_n \subseteq F_N\) for each \(n, m \geq N\). Therefore \(x_n, x_m \in F_N\) for each \(n, m \geq N\) and thus, we have
\[
p(x_n, x_m) - p(x_m, x_m) \leq D(F_n) < \epsilon.
\]
By Theorem 2.10, \(\langle x_n \rangle\) is Cauchy sequence. Since \(X\) is complete, there exists \(x^* \in X\) such that \(p(x_n, x^*) - p(x^*, x^*) < \epsilon\) for each \(n \geq N\).

Let \(n = N\) be fixed. Then the subsequence \(\{x_n, x_{n+1}, \ldots\}\) of the sequence \(\langle x_n \rangle\) is contained in \(F_n\), and still converges to \(x^*\). \(F_n\) is closed in complete metric partial space \((X,p)\), it is complete and so \(x^* \in F_n\) for each \(n \in \mathbb{N}\). Hence \(x^* \in \cap_{n=1}^{\infty} F_n\), that is \(\cap_{n=1}^{\infty} F_n \neq \emptyset\).

Finally, we show that \(x^*\) is unique in \(\cap_{n=1}^{\infty} F_n\). If \(y \in \cap_{n=1}^{\infty} F_n\), then \(x^*, y \in F_n\) for each \(n \in \mathbb{N}\). Therefore \(0 \leq p(x^*, y) - p(y, y) \leq D(F_n) \rightarrow 0\) as \(n \rightarrow \infty\) and \(0 \leq p(x^*, y) - p(x^*, x^*) \leq D(F_n) \rightarrow 0\) as \(n \rightarrow \infty\). Thus \(p(x^*, y) = p(y, y) = p(x^*, x^*)\). By (P2), \(x^* = y\). \(\square\)

**Definition 2.14 ([8]).** Let \((X,p)\) be a metric partial space. The mapping \(f : X \rightarrow X\) is called a contraction on \(X\) if there exists \(k \in (0,1)\) such that for every \(x, y \in X\) we have
\[
(2.1) \quad p(f(x), f(y)) - p(f(y), f(y)) \leq k(p(x, y) - p(y, y)).
\]

**Theorem 2.15 ([8]).** Each contraction in a complete metric partial space has a unique fixed point.

**Proof.** Suppose \(f : X \rightarrow X\) is contraction in a complete metric partial space. Let
\[
x_{n+1} = f(x_n) \quad \text{for } n \geq 0.
\]
We will first show that \(\langle x_n \rangle\) is a Cauchy sequence.

Since \(f\) is contraction we obtain
\[
\begin{align*}
p(f(x_0), f(x_1)) - p(f(x_1), f(x_1)) & \leq k(p(x_0, x_1) - p(x_1, x_1)) \\
p(f(x_1), f(x_2)) - p(f(x_2), f(x_2)) & \leq k^2(p(x_0, x_1) - p(x_1, x_1)) \\
\vdots \\
p(f(x_n), f(x_{n+1})) - p(f(x_{n+1}), f(x_{n+1})) & \leq k^{n+1}(p(x_0, x_1) - p(x_1, x_1)).
\end{align*}
\]
For all $n, m \in \mathbb{N}$ we obtain
\[
p(f(x_n), f(x_{n+m})) - p(f(x_{n+m}), f(x_{n+m})) \leq p(f(x_n), f(x_{n+m}))
\]
\[
+ p(f(x_{n+m-1}), f(x_{n+m})) - p(f(x_{n+m-1}), f(x_{n+m-1})) - p(f(x_{n+m}), f(x_{n+m}))
\]
\[
\leq k^{n+m-1}(p(x_0, x_1) - p(x_1, x_1)) + p(f(x_{n+m-1}), f(x_{n+m}))
\]
\[
- p(f(x_{n+m}), f(x_{n+m}))
\]
\[
\leq k^{n+m-1}(p(x_0, x_1) - p(x_1, x_1)) + p(f(x_{n+m-1}), f(x_{n+m-2}))
\]
\[
+ p(f(x_{n+m-2}), f(x_{n+m})) - p(f(x_{n+m-2}), f(x_{n+m-2}))
\]
\[
- p(f(x_{n+m}), f(x_{n+m}))
\]
\[
\leq (k^{n+m-1} + k^{n+m-2} + \ldots + k^n)(p(x_0, x_1) - p(x_1, x_1))
\]
\[
= \frac{k^n}{1-k}(p(x_0, x_1) - p(x_1, x_1)).
\]
By Theorem 2.10, $(x_n)$ to be a Cauchy sequence. Since $(X, p)$ is a complete partial metric space, $(x_n)$ properly converges to $x^* \in X$ say.

We now show that $x^*$ is a fixed point of $f$. For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$,
\[
p(x_n, x^*) - p(x_n, x_n) < \frac{\epsilon}{1+k}
\]
and
\[
p(x_n, x^*) - p(x^*, x^*) < \frac{\epsilon}{1+k},
\]
Thus for $n \geq N$,
\[
p(f(x^*), x^*) - p(x^*, x^*) \leq p(f(x^*), f(x_n)) + p(f(x_n), x^*) - p(f(x_n), f(x_n))
\]
\[
- p(x^*, x^*)
\]
\[
\leq (p(f(x_n), x^*) - p(f(x_n), f(x_n))) + (p(f(x^*), f(x_n))
\]
\[
- p(x^*, x^*)
\]
\[
\leq k(p(x_n, x^*) - p(x_n, x_n) + k(p(x^*, x_n) - p(x^*, x^*)
\]
\[
< k\left(\frac{\epsilon}{1+k} + \frac{\epsilon}{1+k}\right)
\]
\[
< \epsilon.
\]
Thus, as $\epsilon > 0$ arbitrary, then
\[
p(f(x^*), x^*) = p(x^*, x^*)
\]
Similarly, for $n \geq N$,
\[
p(f(x^*), x^*) - p(f(x^*), f(x^*)) \leq p(f(x^*), f(x_n)) + p(f(x_n), x^*) - p(f(x_n), f(x_n))
\]
\[
- p(f(x^*), f(x^*))
\]
\[
\leq (p(f(x_n), x^*) - p(f(x_n), f(x_n))) + (p(f(x^*), f(x_n))
\]
\[
- p(f(x^*), f(x^*))
\]
\[
\leq k(p(x_n, x^*) - p(x_n, x_n) + k(p(x^*, x_n) - p(x^*, x^*)
\]
\[
< k\left(\frac{\epsilon}{1+k} + \frac{\epsilon}{1+k}\right)
\]
\[
< \epsilon.
\]
Thus, as \( \epsilon > 0 \) arbitrary, then
\[
(2.3) \quad p(f(x^*), x^*) = p(f(x^*), f(x^*))
\]
Thus from (2.2),(2.3) and by (P2), \( f(x^*) = x^* \) and so \( f \) has been that shown to have a fixed point.
It just remains to show that \( x^* \) is unique.
Suppose \( y^* \in X \) and \( y^* = f(y^*) \), then,
\[
p(x^*, y^*) - p(y^*, y^*) = p(f(x^*), f(y^*)) - p(f(y^*), p(y^*)) 
\leq k(p(x^*, y^*) - p(y^*, y^*)).
\]
it follow \( p(x^*, y^*) - p(y^*, y^*) = 0 \) as \( 0 \leq k < 1 \).
Similarly, Suppose \( x^* \in X \) and \( x^* = f(x^*) \), then,
\[
p(x^*, y^*) - p(x^*, x^*) = p(f(x^*), f(y^*)) - p(f(x^*), f(x^*)) 
\leq k(p(x^*, y^*) - p(x^*, x^*)).
\]
it follow \( p(x^*, y^*) - p(x^*, x^*) = 0 \) as \( 0 \leq k < 1 \).
According By axiom (P2), \( p(x^*, x^*) = p(x^*, x^*) = p(y^*, y^*) \), thus \( y^* = x^* \) is unique. \( \square \)

3. **Main results**

In this main result will be shown a fixed point theorem of both Caristi [4] and Banach [8] on partial metric space. In addition it shall be shown the prove Caristi fixed point theorem with two methods, that is, without and use Ekeland’s variational principle. Similarly, for the Banach fixed point theorem.

We start with the following lemma needed to prove our main result.

**Lemma 3.1.** Let \( (X, p) \) be a partial metric space and the function \( \varphi : X \rightarrow [0, \infty) \) is lower semicontinuous. For any \( x, y \in X \) we define relation ” \( \preceq_p \) ” on \( X \) by
\[
(3.1) \quad x \preceq_p y \iff p(x, y) - p(x, x) \leq \varphi(x) - \varphi(y)
\]
Then the relation ” \( \preceq_p \) ” is partial ordered on \( X \).

**Proof.** (i) It is clear that \( p(x, x) - p(x, x) = 0 = \varphi(x) - \varphi(x) \) so that \( x \preceq_p x \) is reflexif.
(ii) If \( x \preceq_p y \) then \( p(x, y) - p(x, x) \leq \varphi(x) - \varphi(y) \) and if \( y \preceq_p x \) then \( p(y, x) - p(y, y) \leq \varphi(y) - \varphi(x) \). This implies \( 2p(x, y) - p(x, x) - p(y, y) = 0 \). Of course \( p(x, y) = p(x, x) = p(y, y) \). By P2, we obtain \( x = y \).
(iii) If \( x \preceq_p y \) then \( p(x, y) - p(x, x) \leq \varphi(x) - \varphi(y) \) and if \( y \preceq_p z \) then \( p(y, z) - p(y, y) \leq \varphi(y) - \varphi(z) \). This implies \( 2p(x, z) - p(x, x) \leq p(x, y) + p(y, z) - p(y, y) - p(x, x) \leq \varphi(x) - \varphi(z) \) and hence \( x \preceq_p z \). \( \square \)

**Lemma 3.2.** *(Zorn’s Lemma)*. Let \( X \) be nonempty partially ordered set in which every totally set has a upper bound. Then \( X \) has at least one maximal element.

The following is Caristi fixed point theorem type on partial metric space.
Theorem 3.3. Let \((X, p)\) be a complete partial metric space and \(f : X \to X\) be a mapping on \(X\). Suppose there exists a lower semicontinuous function \(\varphi : X \to [0, \infty)\) such that
\[
p(x, f(x)) - p(f(x), f(x)) \leq \varphi(x) - \varphi(f(x))
\]
for all \(x \in X\). Then \(f\) has a fixed point.

Proof. For any \(x, y \in X\) we define the relation "\(\preceq_p\)" on \(X\) by
\[
x \preceq_p y \iff p(x, y) - p(y, y) \leq \varphi(x) - \varphi(y).
\]
By Lemma 3.1, \((X, \preceq_p)\) is a partial ordered. Let \(x_0 \in X\) be an arbitrary but fixed element of \(X\). Then by Zorn’s Lemma, we obtain totally ordered subset \(M\) of \(X\) containing \(x_0\).

Let \(M = \{x_\alpha\}_{\alpha \in I}\), where \(I\) is totally ordered and
\[
x_\alpha \preceq_p x_\beta \iff \alpha \preceq_p \beta
\]
for all \(\alpha, \beta \in I\).

Since \(\{\varphi(x_\alpha)\}\) is decreasing in \(\mathbb{R}^+\), there exists \(r \geq 0\) such that \(\varphi(x_\alpha) \to r\) as \(\alpha\) increases.

Let \(\epsilon > 0\) be given. Then there exists \(\alpha_0 \in I\) such that for \(\alpha \geq \alpha_0\) we have
\[
r \leq \varphi(x_\alpha) \leq \varphi(x_{\alpha_0}) \leq r + \epsilon.
\]

Let \(\beta \geq \alpha \geq \alpha_0\), then by (4) we obtain
\[
p(x_\alpha, x_\beta) - p(x_\beta, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta) \leq r + \epsilon - r = \epsilon.
\]
which implies that \(\{x_\alpha\}\) is a Cauchy net in \(X\) by Theorem 2.10. Since \(X\) is complete, there exists \(x \in X\) such that \(x_\alpha \to x\) as \(\alpha\) increases. From the lower semicontinuity of \(\varphi\) we deduce that \(\varphi(x_{\alpha_0}) \leq r\). If \(\beta \geq \alpha\) then \(p(x_\alpha, x_\beta) - p(x_\beta, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta)\).

Letting \(\beta\) as increases we obtain
\[
p(x_\alpha, x) - p(x, x) \leq \varphi(x_\alpha) - r \leq \varphi(x_\alpha) - \varphi(x).
\]
which gives is \(x_\alpha \preceq_p x\) for \(\alpha \in I\). In particular, \(x_0 \preceq_p x\). Since \(M\) is maximal, \(x \in M\). Moreover, the condition (3) implies that
\[
x_\alpha \preceq_p x \preceq_p f(x) \quad \text{for all} \quad \alpha \in I.
\]
Again by maximality, \(f(x) \in M\). Since \(x \in M\), \(f(x) \preceq_p x\) and hence \(f(x) = x\). \(\square\)

The mapping \(f\) satisfying (3.2) is called Caristi’s maps. Again we write self map \(f : X \to X\) contraction on a partial metric space, the following
\[
p(f(x), f(y)) - p(f(x), f(x)) \leq kk[p(x, y) - p(y, y)]
\]
for all \(x, y \in X\) and for some \(k \in (0, 1)\)

For \(y = f(x)\) will deduce the following
\[
p(f(x), f^2(x)) - p(f(x), f(x)) \leq k[p(x, f(x)) - p(f(x), f(x))]
\]
thus
\[
[p(x, f(x)) - p(x, x)] - k[p(x, f(x)) - p(x, x)]
\]
\[
\leq [p(x, f(x)) - p(x, x)] - [p(f(x), f^2(x)) - p(f(x), f(x))].
\]
Hence
\[
(1-k)[p(x, f(x)) - p(x, x)] \\
\leq [p(x, f(x)) - p(x, x)] - [p(f(x), f^2(x)) - p(f(x), f(x))]
\]
or
\[
p(x, f(x)) - p(x, x) \\
\leq \frac{1}{1-k}[p(x, f(x)) - p(x, x)] - \frac{1}{1-k}[p(f(x), f^2(x)) - p(f(x), f(x))]
\]
If the function \( \varphi : X \rightarrow [0, \infty) \) defined by
\[
\varphi(x) = \frac{1}{1-k}[p(x, f(x)) - p(x, x)],
\]
we obtained
\[
p(x, f(x)) - p(x, x) \leq \varphi(x) - \varphi(f(x)).
\]
It appears that \( f \) is a Caristi’s mapping on partial metric space. Thus the contraction mapping is a special case of Caristi’s mapping.

The following will be given the Ekeland’s Variational Principle on partial metric spaces.

**Theorem 3.4.** Let \((X, p)\) be a complete partial metric space and \( \varphi : X \rightarrow [0, \infty) \) be a lower semicontinuous function. Let \( \epsilon > 0 \) and \( \bar{x} \in X \) be given such that
\[
\varphi(\bar{x}) \leq \inf_{x \in X} \varphi(x) + \epsilon.
\]
Then for a given \( \delta > 0 \) there exists \( x^* \in X \) such that
\begin{itemize}
  \item[(a)] \( \varphi(x^*) \leq \varphi(\bar{x}) \)
  \item[(b)] \( p(\bar{x}, x^*) \leq \delta + p(x^*, x^*) \)
  \item[(c)] \( \varphi(x^*) \leq \varphi(x) + \frac{\epsilon}{\delta}(p(x, x^*) - p(x^*, x^*)) \) for all \( x \in X \setminus \{x^*\} \).
\end{itemize}

**Proof.** For \( \delta > 0 \), we set \( p_\delta(x, y) = \frac{1}{\delta}(p(x, y) - p(y, y)) \). Then \( p_\delta \) is equivalent to \( p \) and \((X, p_\delta)\) is complete. Let us define a partial ordering \( \preceq \) on \( X \) by
\[
(3.9) \quad x \preceq y \iff \varphi(x) \leq \varphi(y) - \epsilon(p(x, y) - p(y, y)).
\]
It is easy to see that this ordering is (i) reflexive, that is, for all \( x \in X, x \preceq x \); (ii) antisymmetric, that is, for all \( x, y \in X, x \preceq y \) and \( y \preceq x \) imply \( x = y \); (iii) transitive, that is, for all \( x, y, z \in X, x \preceq y \) and \( y \preceq z \) imply \( x \preceq z \).

We define a sequence \( \langle E_n \rangle \) of subsets of \( X \) as follow: We start \( x_1 = \bar{x} \) and define
\[
E_1 = \{ x \in X : x \preceq x_1 \}; \quad x_2 \in E_1 \text{ such that } \varphi(x_2) \leq \inf_{x \in E_1} \varphi(x) + \frac{\epsilon}{\delta},
\]
\[
E_2 = \{ x \in X : x \preceq x_2 \}; \quad x_3 \in E_2 \text{ such that } \varphi(x_3) \leq \inf_{x \in E_2} \varphi(x) + \frac{\epsilon}{\delta^2},
\]
and inductively
\[
E_n = \{ x \in X : x \preceq x_n \}; \quad x_{n+1} \in E_n \text{ such that } \varphi(x_{n+1}) \leq \inf_{x \in E_n} \varphi(x) + \frac{\epsilon}{\delta^n}.
\]
Clearly, \( E_1 \supset E_2 \supset E_3 \cdots \). Let \( u_n \in E_n \) with \( u_m \rightarrow u \in X \). Then \( u_m \preceq x_n \) and
so \( \varphi(u_m) \leq \varphi(x_n) - \epsilon (p_\delta(u_m, x_n) - p_\delta(x_n, x_n)) \). therefore

\[
\varphi(u) \leq \lim_{m \to \infty} \inf \varphi(u_m) \\
\leq \varphi(x_n) - \epsilon \lim_{m \to \infty} \inf(p_\delta(u_m, x_n) - p_\delta(x_n, x_n)) \\
\leq \varphi(x_n) - \epsilon (p_\delta(u, x_n) - p_\delta(x_n, x_n)).
\]

Thus \( u \in E_n \). We conclude that each \( E_n \) is closed.

Take any point \( x \in E_n \), one on hand \( x \preceq x_n \), implies that

\[ (3.10) \quad \varphi(x) \leq \varphi(x_n) - \epsilon (p_\delta(x, x_n) - p_\delta(x_n, x_n)). \]

We observe that \( x \) also belongs to \( E_{n-1} \supset E_n \), so it is one of the points which entered in the competition when we picked \( x_n \). Therefore,

\[ (3.11) \quad \varphi(x_n) \leq \inf_{y \in E_{n-1}} \varphi(y) + \frac{\epsilon}{2^{n-1}} \leq \varphi(x) + \frac{\epsilon}{2^{n-1}}. \]

From (3.10) and (3.11), we obtain

\[ \varphi(x) + \epsilon (p_\delta(x, x_n) - p_\delta(x_n, x_n)) \leq \varphi(x) + \frac{\epsilon}{2^{n-1}}. \]

It is follow that

\[ p_\delta(x, x_n) - p_\delta(x_n, x_n) \leq 2^{-n+1}. \]

for all \( x \in E_n \).

Which resulted \( D(E_n) \leq 2^{-n} \) and hence \( D(E_n) \to 0 \) \( (n \to \infty) \).

Since \( (X, p_\delta) \) is complete and \( (E_n) \) is a decreasing sequence of closed sets, by Theorem 2.13, we infer that

\[ \bigcap_{n=1}^{\infty} E_n = \{x^*\} \]

Since \( x^* \in E_1 \), we have

\[ x^* \preceq x_1 = \bar{x} \iff \varphi(x^*) \leq \varphi(\bar{x}) - \epsilon (p_\delta(x^*, \bar{x}) - p_\delta(\bar{x}, \bar{x})) \leq \varphi(\bar{x}). \]

Hence \( (a) \) is proved.

Now we write

\[ p_\delta(\bar{x}, x_n) - p_\delta(x_n, x_n) = p_\delta(x_1, x_n) - p_\delta(x_n, x_n) \]

\[ \leq \sum_{i=1}^{n-1} [p_\delta(x_i, x_{i+1}) - p_\delta(x_i, x_i)] \]

\[ \leq \sum_{i=1}^{n-1} 2^{-i+1}. \]

and taking limit as \( n \to \infty \), we obtain

\[ \frac{1}{\delta} (p(\bar{x}, x^*) - p(x^*, x^*)) = p_\delta(\bar{x}, x^*) - p_\delta(x^*, x^*) \leq 1 \]
and so \( p(\bar{x}, x^*) \leq \delta + p(x^*, x^*) \). This proves (b).

Finally, let \( x \neq x^* \), of course \( x \notin \bigcap_{n=1}^{\infty} E_n \), so \( x \neq x^* \), which means that

\[
\varphi(x) > \varphi(x^*) - \epsilon[p(x, x^*) - p(x, x^*)] \\
= \varphi(x^*) - \frac{\epsilon}{2}[p(x, x^*) - p(x^*, x^*)]
\]

and hence (c) proved. \( \square \)

We now present, so called the weak formulation of Ekeland’s Variational Principle.

**Corollary 3.5.** Let \((X, p)\) be a complete partial metric space and \( \varphi : X \rightarrow [0, \infty) \) be a lower semicontinuous function. Then for any given \( \epsilon > 0 \) there exists \( x^* \in X \) be such that

\[
\varphi(x^*) \leq \inf_{x \in X} \varphi(x) + \epsilon.
\]

and

\[
\varphi(x) < \varphi(x^*) + \epsilon[p(x, x^*) - p(x^*, x^*)].
\]

for all \( x \neq x^* \in X \).

**Definition 3.6.** Let \((X, p)\) be a partial metric space. The function \( f : X \rightarrow X \) is called continuous at the point \( x_0 \) if, for any sequence \( \langle x_n \rangle \) in \( X \) converges to \( x_0 \), then a sequence \( \langle f(x_n) \rangle \) converges to \( f(x_0) \).

**Lemma 3.7.** Let \((X, p)\) be a partial metric space and the function \( f : X \rightarrow X \). Then for each \( x \in X \), the function \( \varphi_x : X \rightarrow [0, \infty) \) defined by

\[
\varphi_x(y) = p(x, f(y)).
\]

If \( f \) is contraction, then the function \( \varphi_x \) is continuous on \( X \).

**Proof.** Assume that a sequence \( y_n \) converges to \( y \) in \( X \), then \( \lim_{n \to \infty} p(y_n, y) = p(y, y) = 0 \). For each \( x \in X \) and \( k \in (0, 1) \) and by (P4), we have

\[
|\varphi_x(y_n) - \varphi_x(y)| = |p(x, f(y_n)) - p(x, f(y))| \\
\leq |p(f(y), f(y_n)) - p(f(y), f(y))| \\
= p(f(y), f(y_n)) - p(f(y), f(y)) \\
< k(p(y_n, y) - p(y, y)).
\]

This yields \( \lim_{n \to \infty} \varphi_x(y_n) = \varphi_x(y) \) because \( \lim_{n \to \infty} p(y_n, y) = p(y, y) = 0 \). \( \square \)

Now, we will present the proved fixed point theorem which using Ekeland’s variational principle.

As first applications of Ekeland’s variational principle, we prove Caristi’s fixed point theorem version.

**Theorem 3.8.** Let \((X, p)\) be a complete partial metric space and \( f : X \rightarrow X \) be a mapping on \( X \). Suppose there exists a lower semicontinuous function \( \varphi : X \rightarrow [0, \infty) \) such that

\[
p(x, f(x)) - p(f(x), f(x)) \leq \varphi(x) - \varphi(f(x))
\]

for all \( x \in X \). Then \( f \) has a fixed point.
Proof. By using Corollary 3.5 with \( \epsilon = 1 \), we obtain \( x^* \in X \) such that
\[
\varphi(x^*) < \varphi(x) + [p(x, x^*) - p(x, x)]
\]
for all \( x \neq x^* \).

We assumed for all \( y = f(x^*) \in X \) are such that \( y \neq x^* \). Then from (3.12) and (3.13), we have
\[
p(x^*, y) - p(y, y) \leq \varphi(x^*) - \varphi(y)
\]
and
\[
\varphi(x^*) < \varphi(y) + [p(y, x^*) - p(y, y)]
\]
which cannot hold simultaneously. Hence, \( x^* = f(x^*) \). \( \square \)

As second applications of Ekeland’s variational principle, we prove the well-known Banach contraction theorem.

Theorem 3.9. Let \((X, p)\) be a complete partial metric space and \( f : X \to X \) be a contraction mapping. Then \( f \) has a unique fixed point in \( X \).

Proof. Consider the function \( \varphi : X \to [0, \infty) \) defined by
\[
\varphi(x) = p(x, f(x)),
\]
for all \( x \in X \).

By Lemma 3.7, \( \varphi \) is a continuous on \( X \). Choose \( \epsilon > 0 \) such that \( 0 < \epsilon < 1 - k \), where \( k \in (0, 1) \). By Lemma 3.5, there exists \( x^* \in X \) such that
\[
\varphi(x^*) < \varphi(x) + \epsilon[p(x, x^*) - p(x, x)]
\]
for all \( x \in X \).

Putting \( x = f(x^*) \), we have
\[
p(x^*, f(x^*)) \leq p(x, f(x)) + \epsilon[p(x, x^*) - p(x, x)]
= p(f(x^*), f(f(x^*))) + \epsilon[p(f(x^*), x^*) - p(f(x^*), f(x^*))]
\leq k[p(x^*, f(x^*)) - p(f(x^*), f(x^*)) + \epsilon[p(f(x^*), x^*) - p(f(x^*), f(x^*))]]
= (k + \epsilon)[p(x^*, f(x^*)) - p(f(x^*), f(x^*))]
\leq (k + \epsilon)p(x^*, f(x^*))
\]

If \( x^* \neq f(x^*) \), then we obtain \( 1 \leq (k + \epsilon) \), which contradict to our assumption that \( 1 > (k + \epsilon) \). Therefore, we have \( x^* = f(x^*) \). The uniqueness of \( x^* \) can be proved as in the Theorem 2.15. \( \square \)

References


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