$p$-Summable Sequence Spaces with 2-Inner Products

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Abstract. We revisit the space $\ell^p$ of $p$-summable sequences of real numbers. In particular, we show that this space is actually contained in a (weighted) 2-inner product space. For $p > 2$, we also obtain a result which describe how the weighted 2-inner product space is associated to the weights.

Key words. 2-Inner product spaces; 2-normed spaces; $p$-summable sequences; weights

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1 Introduction

The inner product spaces have been, up to now, the most useful spaces in practical applications of functional analysis. These spaces were initially introduced by D. Hilbert [4] in 1912. By $\ell^p = \ell^p(\mathbb{R})$ we denote the space of all $p$-summable sequences of real numbers. We know that for $p \neq 2$, the space $\ell^p$ is not an inner product space, since the usual norm $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ on $\ell^p$ does not satisfy the parallelogram law. So, it can not be derived from an inner product for $p \neq 2$. There is a semi-inner product on $\ell^p$ as in [7], but having a semi-inner product is not as nice as having an inner product. Gunawan et al. [2] have defined a new norm satisfying the parallelogram law and thus it can be derived from an inner product on $\ell^p$. We also know that the space of $p$-summable sequences of real numbers $\ell^p$, which is equipped with usual 2-norm $\|x\|_p$ defined by Gunawan [3], also is not a 2-inner product space with $p \neq 2$. Because the 2-norm $\|x\|_p$ does not satisfy the parallelogram law. Taking inspiration from [2], one question arises: can we define a 2-norm on $\ell^p$ which satisfies the parallelogram law? The reason why we are interested in the parallelogram law is because we eventually wish to define a 2-inner
product, possibly with weights, on $\ell^p$, so that we can define orthogonality and many other notions on this space.

In this paper, we have obtained a different 2-norm $\|\cdot, \cdot\|_{2,w}$, which is not equivalent to the usual 2-norm $\|\cdot, \cdot\|_p$ on $\ell^p$ (except with the condition $p \neq 2$), however, it satisfies the parallelogram law. For $p > 2$, we also obtain a result which describe how the weighted 2-inner product space is associated to the weights. We discuss the properties of the induced 2-norm and its relationship with the usual 2-norm on $\ell^p$. We also find that the 2-inner product is actually defined on a larger space.

2 Definitions and Preliminaries

Let $X$ be a real vector space of dimension $d \geq 2$. The real-valued function $\langle \cdot, \cdot | \cdot \rangle$ which satisfies the following properties on $X^3$ is called 2-inner product on $X$, and the pair $(X, \langle \cdot, \cdot | \cdot \rangle)$ is called a 2-inner product space:

1. $\langle x, x | z \rangle \geq 0$; $\langle x, x | z \rangle = 0$ if and only if $x$ and $z$ are linearly dependent,
2. $\langle x, y | z \rangle = \langle y, x | z \rangle$,
3. $\langle x, x | z \rangle = \langle z, z | x \rangle$,
4. $\langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle$, for $\alpha \in \mathbb{R}$,
5. $\langle x_1 + x_2, y | z \rangle = \langle x_1, y | z \rangle + \langle x_2, y | z \rangle$.

The function $\|\cdot, \cdot\|$, which follows four properties, is called a 2-norm and the pair $(X, \|\cdot, \cdot\|)$ is called a 2-normed space:

1. $\|x, z\| \geq 0$, for $x, z \in X$, $\|x, z\| = 0$ if and only if $x$ and $z$ are linearly dependent,
2. $\|x, z\| = \|z, x\|$, for $x, z \in X$,
3. $\|\alpha x, z\| = |\alpha| \|x, z\|$, for $x, y \in X$ and $\alpha \in \mathbb{R}$,
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for $x, y, z \in X$.

A sequence $x = (x_j)$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is called a Cauchy sequence, if there are $y$ and $z$ in $X$ such that $y$ and $z$ are linearly independent, $\lim_{i, j \to \infty} \|x_i - x_j, y\| = 0$ and $\lim_{i, j \to \infty} \|x_i - x_j, z\| = 0$.

A sequence $x = (x_j)$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is called a convergent sequence, if there is an $\xi$ in $X$ such that $\lim_{j \to \infty} \|x_j - \xi, z\| = 0$ for every nonzero $z$ in $X$.

As we work with sequence spaces of real numbers, we will use the sum notation $\sum_k$ instead of $\sum_{k=1}^\infty$, for brevity.
We know that $\ell^p$ ($1 \leq p \leq \infty$) can be equipped with usual 2-norm $\|\cdot\|_p$:

$$
\|x_1, x_2\|_p := \left( \frac{1}{2} \sum_{k_1} \sum_{k_2} \left\| \begin{array}{cc} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{array} \right\|_p^p \right)^{\frac{1}{p}},
$$

(2.1)

where $x = (x_{1k})$ and $x = (x_{2k})$. For $p = \infty$, the formula reduces to

$$
\|x_1, x_2\|_\infty := \sup_{k_1} \sup_{k_2} \left\| \begin{array}{cc} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{array} \right\|_{\infty},
$$

defined by Gunawan [3]. The 2-norm given by (2.1) does not satisfy the parallellogram law with $p \neq 2$.

To show this we can give an example: If we take $x_1 = (1, 0, 0, ...), y_1 = (0, 1, 0, ...), x_2 = (0, ..., 0, 1, 0, ...)$ (1 is in the $k_0^{th}$ term) in $\ell^p$, then we have $\|x_1, x_2\|_p = 1, \|y_1, x_2\|_p = 1, \|x_1 + y_1, x_2\|_p = \|x_1 - y_1, x_2\|_p = 2^{1/p}$. Hence,

$$2\|x_1, x_2\|_p^2 + 2\|y_1, x_2\|_p^2 \neq \|x_1 - y_1, x_2\|_p^2 + \|x_1 + y_1, x_2\|_p^2.$$

We can define a new 2-norm on $\ell^p$

$$
\|x_1, x_2\|_* := \left( \frac{1}{2} \sum_{k_1} \sum_{k_2} \left\| \begin{array}{cc} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{array} \right\|_{\infty}^p \right)^{\frac{1}{p}},
$$

which is not equivalent to $\|\cdot\|_p$, and also does not satisfy the parallellogram law. To show these, we can give the following examples, respectively:

It is easy to see that $\|x, y\|_* \leq \|x, y\|_p$. But we don’t have a constant $D > 0$ such that $\|x, y\|_* \geq D\|x, y\|_p$ for every $x, y \in \ell^p$. Take $x = (1, 0, 0, ...)$ and $y = (0, ..., 0, 1, 0, ...)$ (1 is in the $k_0^{th}$ term), and compute $\|x, y\|_p = 1$ and $\|x, y\|_* = \frac{1}{k_0}$. If $k_0 \to \infty$ then $\|x, y\|_* \to 0$. So, there is not a constant $D > 0$ such that $\|x, y\|_* \geq D\|x, y\|_p$.

Take $x = (1, 0, 0, ...), y = (0, 1, 0, ...), z = (0, 0, 1, 0, ...)$ and compute $\|x, z\|_* = \frac{1}{3}, \|y, z\|_* = \frac{1}{6}$ and $\|x + y, z\|_* = \|x - y, z\|_* = (\frac{1}{3p} + \frac{1}{6q})^{\frac{1}{2}}$. Then

$$2\|x, z\|_*^2 + 2\|y, z\|_*^2 \neq \|x + y, z\|_*^2 + \|x - y, z\|_*^2.$$

One question arises: can we define a 2-norm on $\ell^p$ which satisfies the parallellogram law? In the next sections we will give the answer of this question.

3 Results for $1 \leq p \leq 2$

In this section, we let $1 \leq p \leq 2$, unless otherwise stated. First, we observe from [2] that $\ell^p \subseteq \ell^2$ (as sets). With respect to the 2-norms on these spaces, we have:

**Theorem 3.1** If $x, z \in \ell^p$ with $\|x, z\|_p < \infty$, then $x, z \in \ell^2$ with $\|x, z\|_2 < \infty$. 
Proof. Let \( x, z \in \ell^p \) with \( \| x, z \|_p < \infty \). Since \( \ell^p \subseteq \ell^2 \), see in [2], then \( x, z \in \ell^2 \). Moreover,

\[
\| x, z \|_2^2 = \frac{1}{2} \sum_{k_1} \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right|^2 = \frac{1}{2} \sum_{k_1} \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right|^{2-p} \| x_{k_1} x_{k_2} \|^p
\]

\[
\leq \sup_{k_1 \succ k_2} \sup_{k_2} \left( \sum_{k_1 \succ k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right| \right)^{\frac{2-p}{p}} \sum_{k_1 \succ k_2} \left( \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right|^p \right)^{\frac{1}{p}}
\]

\[
= \left( \sum_{k_1 \succ k_2} \left( \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right| \right)^{\frac{1}{p}} \| x_{k_1} x_{k_2} \|^p \right)^{\frac{2}{p}} \leq \left( \sum_{k_1 \succ k_2} \left( \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right|^p \right)^{\frac{1}{p}} \right)^{\frac{2}{p}}
\]

Taking the square roots of both sides, we get

\[
\left( \frac{1}{2} \sum_{k_1} \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{2} \sum_{k_1} \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right|^p \right)^{\frac{1}{p}}
\]

which means that \( x, z \in \ell^2 \) with \( \| x, z \|_2 < \infty \).

Thus we realize that \( \ell^p \) is a subspace of \( (\ell^2, \|, \|) \). Hence \( \ell^p \) can be equipped with the 2-inner product

\[
\langle x, y \mid z \rangle = \frac{1}{2} \sum_{k_1} \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right| \left| \frac{y_{k_1} y_{k_2}}{z_{k_1} z_{k_2}} \right|
\]

and the 2-norm

\[
\| x, z \|_2 = \left( \frac{1}{2} \sum_{k_1} \sum_{k_2} \left| \frac{x_{k_1} x_{k_2}}{z_{k_1} z_{k_2}} \right|^2 \right)^{\frac{1}{2}}
\]

Being an induced 2-norm from the 2-inner product, the 2-norm \( \|, \|_2 \) of course satisfies the parallelogram law:

\[
\| x + y, z \|_2^2 + \| x - y, z \|_2^2 = 2 \| x, z \|_2^2 + 2 \| y, z \|_2^2
\]

for every \( x, y, z \in \ell^p \). We can check that as a subspace of \( (\ell^2, \|, \|) \), \( \ell^p \) is not closed, later.

A more general result is formulated in the following proposition, which describes the monotonicity property of the 2-norms on \( \ell^p \) spaces.
Proposition 3.2 1. Let \( 1 \leq p \leq q \leq \infty \). If \( x, z \in \ell^p \), then \( x, z \in \ell^q \) with \( \|x, z\|_q \leq \|x, z\|_p \).

2. The converse is not true for \( 1 \leq p < q \leq \infty \).

Proof.

1. Let \( 1 \leq p \leq q \leq \infty \) and \( x, z \in \ell^p \). Since \( \ell^p \subseteq \ell^q \) (see in Proposition 2.1, [2]) then \( x, z \in \ell^q \).

Moreover we have

\[
\|x, z\|^q_q = \frac{1}{2} \sum_{k_1, k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^q = \frac{1}{2} \sum_{k_1, k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^{q-p} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^p
\]

\[
= \sum_{k_1 > k_2} \sum_{k_1 > k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^{q-p} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^p
\]

\[
\leq \sup_{k_1 > k_2} \sup_{k_1 > k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^{q-p} \sum_{k_1 > k_2} \sum_{k_1 > k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^p
\]

\[
\leq \left( \sum_{k_1 > k_2} \sum_{k_1 > k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^{q-p} \right)^{\frac{q-p}{q}} \left( \sum_{k_1 > k_2} \sum_{k_1 > k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^p \right)^{\frac{1}{q}}
\]

\[
= \left( \frac{1}{2} \sum_{k_1, k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^p \right)^{\frac{q}{p}}.
\]

Taking the \( q \)-th roots of both sides, we get \( \|x, z\|^q_q \leq \|x, z\|_p \).

2. To show that the converse is not true for \( 1 \leq p < q \leq \infty \), that is; to show that for every \( x, z \in \ell^q \) there exist \( x \notin \ell^p \) or \( z \notin \ell^p \) with \( \|x, z\|_q < \|x, z\|_p \), one may take \( x = (x_k) = \left( \frac{1}{k^{1/p}} \right)_{k \in \mathbb{N}} \) where \( 1 \leq p < \infty \) and \( z = (1, 0, ...) \). Then

\[
\|x, z\|^p_p = \left( \frac{1}{2} \sum_{k_1, k_2} \|x_{k_1}, x_{k_2}, z_{k_1}, z_{k_2}\|^p \right)^{\frac{1}{p}} = \left( \frac{1}{2} \sum_{k_1, k_2} \frac{|z_{k_2}|}{|k_1^{1/p}|} - \frac{|z_{k_1}|}{|k_2^{1/p}|} \right)^{\frac{1}{p}}
\]

\[
= \left( \frac{1}{2} \sum_{k_1} \frac{1}{k_1^{2/p}} + \frac{1}{2} \sum_{k_2} \frac{1}{k_2^{2/p}} \right)^{\frac{1}{p}} = \left( \sum_{k} \frac{1}{k^{2/p}} \right)^{\frac{1}{p}} < \infty,
\]

since \( \frac{q}{p} > 1 \).
When we equip \( \ell^p \) for \( 1 \leq p < 2 \), with \( \| \cdot \|_2 \), one might ask whether \( \| \cdot \|_2 \) is equivalent with the usual 2-norm \( \| \cdot \|_p \). The answer is negative. We already have \( \| x, z \|_2 \leq \| x, z \|_p \) for every \( x, z \in \ell^p \).

The following proposition tells us that we can not control \( \| x, z \|_p \) by \( \| x, z \|_2 \) for every \( x, z \in \ell^p \).

**Proposition 3.3** Let \( 1 \leq p < 2 \). There is no constant \( C > 0 \) such that \( \| x, z \|_2 \geq C \| x, z \|_p \) for every \( x, z \in \ell^p \).

**Proof.** Let \( \{ z_1, z_2 \} \) be a linearly independent set where \( z_1 = (1, 0, ...) \) and \( z_2 = (0, 1, 0, ...) \). For each \( n \in \mathbb{N} \), take \( x^{(n)} = \left( \frac{1}{k^{1/p}} \right) \). Then for \( i = 1, 2 \)

\[
\| x^{(n)}, z_i \|_2^2 = \frac{1}{2} \sum_{k_1} \sum_{k_2} \left| x^{(n)}_{k_1} z_{k_1} \right|^2 = \sum_k \left( \frac{1}{k^{1/p}} \right)^2 = \sum_k \frac{1}{k^{2/p}}
\]

\[
\leq \sum_k \frac{1}{k^2} < \infty \quad \text{(independent of } n \text{)}
\]

while

\[
\| x^{(n)}, z_i \|_p^p = \sum_k \frac{1}{k^{1/p}} < \infty \quad \text{(dependent of } n \text{)}
\]

which tends to \( \infty \) as \( n \to \infty \). Hence

\[
\frac{\| x^{(n)}, z_i \|_2}{\| x^{(n)}, z_i \|_p} \to 0
\]

as \( n \to \infty \). So, there is no constant \( C > 0 \) such that \( \| x, z \|_2 \geq C \| x, z \|_p \) for every \( x, z \in \ell^p \).

**Proposition 3.4** \( \ell^p \) is not closed but dense in \( (\ell^2, \| \cdot \|_2) \).

**Proof.** Let \( \{ a, b \} \) be a linearly independent set where \( a = e_1 = (1, 0, ...) \) and \( b = e_2 = (0, 1, 0, ...) \).

To show that \( \ell^p \) is not closed in \( (\ell^2, \| \cdot \|_2) \) for each \( n \in \mathbb{N} \), we take \( x^{(n)} = \left( \frac{1}{k^{1/p}} \right) \) in \( \| \cdot \|_2 \) with respect to the set \( \{ a, b \} \) such that \( \| x^{(n)} - x, a \|_2 \to 0 \), as \( n \to \infty \) and \( \| x^{(n)} - x, b \|_2 \to 0 \), as \( n \to \infty \). But \( x^{(n)} \in \ell^p \) for each \( n \in \mathbb{N} \), while \( x \notin \ell^p \). Therefore \( \ell^p \) is not closed in \( (\ell^2, \| \cdot \|_2) \).

To show that \( \ell^p \) is dense in \( (\ell^2, \| \cdot \|_2) \), we observe that every \( x = (x_k) \in \ell^2 \) can be approximated arbitrarily close by \( x^{(n)} = (x_1, x_2, ..., x_n, 0, 0, ...) \in \ell^p \), \( n \in \mathbb{N} \) with \( \| x^{(n)} - x, a \|_2 \to 0 \) as \( n \to \infty \) and \( \| x^{(n)} - x, b \|_2 \to 0 \) as \( n \to \infty \).

In this section, we have studied the case where \( 1 \leq p < 2 \). We shall discuss what happens when \( p > 2 \) in the next section.
4 Results for $2 < p < \infty$

Throughout this section, we let $2 < p < \infty$, unless otherwise stated. As we have seen in Proposition 2.1 in [2] and Proposition 3.2, the space $\ell^p$ is now larger than $\ell^2$. Thus the usual 2-inner product and 2-norm on $\ell^2$ are not defined for all sequences in $\ell^p$.

To define an inner product and a new norm on $\ell^p$ which satisfies the parallelogram law, we have to use weights. Let us choose $v = (v_k) \in \ell^p$ where $v_k > 0$, $k \in \mathbb{N}$. Next we define the mapping $(\cdot | \cdot)_v$ which maps every triple of sequences $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ to

$$
\langle x, y | z \rangle_v := \frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} ||x_{k_1} z_{k_1} y_{k_1} z_{k_1} x_{k_2} z_{k_2} y_{k_2} z_{k_2}|| 
$$

and the mapping $\| \cdot \|_{2,v}$ which maps every sequence $x = (x_k)$ and $z = (z_k)$ to

$$
\|x, z\|_{2,v} := \sqrt{\langle x, x | z, z \rangle_v} := \left[ \frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} ||x_{k_1} z_{k_1} y_{k_1} z_{k_1} x_{k_2} z_{k_2} y_{k_2} z_{k_2}|| \right]^{\frac{1}{2}}. 
$$

We observe that the mappings are well defined on $\ell^p$. Indeed, for $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ in $\ell^p$, it follows from Hölder’s inequality that

$$
\frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} ||x_{k_1} z_{k_1} y_{k_1} z_{k_1} x_{k_2} z_{k_2} y_{k_2} z_{k_2}|| 
\leq \left[ \frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p} \right]^{\frac{p-2}{p}} \left[ \frac{1}{2} \sum_{k_1} \sum_{k_2} ||x_{k_1} z_{k_1} y_{k_1} z_{k_1} x_{k_2} z_{k_2} y_{k_2} z_{k_2}|| \right]^{\frac{1}{p}} 
\leq \left( \frac{1}{2} \right)^{\frac{p-2}{p}} \left( \sum_{k_1} |v_{k_1}|^{p} \right)^{\frac{p-2}{p}} \left( \sum_{k_2} |v_{k_2}|^{p} \right)^{\frac{p-2}{p}} \left[ \frac{1}{2} \sum_{k_1} \sum_{k_2} ||x_{k_1} z_{k_1} y_{k_1} z_{k_1} x_{k_2} z_{k_2} y_{k_2} z_{k_2}|| \right]^{\frac{1}{p}}. 
$$

Thus the two mappings are defined on $\ell^p$. Moreover, we have the following proposition, whose proof is left to the reader.

**Proposition 4.1** The mappings in (4.1) and (4.2) define a weighted 2-inner product and a weighted 2-norm, respectively, on $\ell^p$.

We observe that the equation (4.1) can be rewritten as

$$
\langle x, y | z \rangle_v := \frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} ||x_{k_1} z_{k_1} y_{k_1} z_{k_1} x_{k_2} z_{k_2} y_{k_2} z_{k_2}|| 
\leq \sum_{k} |v_k|^{p-2} x_k y_k \sum_{k} |v_k|^{p-2} x_k z_k 
\leq \langle x, y \rangle_v \langle x, z \rangle_v. 
$$
where \( \langle x, y \rangle_v := \sum_k |v_k|^p x_k y_k \) (see, [2]). Thus \( \|\cdot\|_{2,v} \) is a standard 2-norm on \( \ell^p \).

From (4.3), we see that the following inequality

\[
\|x, z\|_{2,v} \leq 2^{\frac{1}{p}} \|v\|_p \|x, z\|_p
\]

holds for every \( x, z \in \ell^p \). It is then tempting to ask whether the two 2-norms are equivalent on \( \ell^p \). The answer, however, is negative, due to the following result.

**Proposition 4.2** There is no constant \( C = C_v > 0 \) such that

\[
\|x, z\|_p \leq C \|x, z\|_{2,v}
\]

for every \( x, z \in \ell^p \).

**Proof.** Let \( \{z_1, z_2\} \) be a linearly independent set where \( z_1 = (1, 0, \ldots) \) and \( z_2 = (0, 1, 0, \ldots) \). Suppose that such a constant exists. Then, for \( x := e_n = (0, \ldots, 0, 1, 0, \ldots) \), where the 1 is the \( n \)th term, we have

\[
1 \leq C |v_i v_n|^\frac{p-2}{p}
\]

for each \( n \in \mathbb{N} \) and for each \( i = 1, 2 \), since \( \|x, z_i\|_p = 1 \) and \( \|x, z_i\|_{2,v} = |v_i v_n|^\frac{p-2}{p} \). But this cannot be true, since \( v_n \to 0 \) as \( n \to \infty \).

According to Proposition 4.2, it is possible for us to find a sequence in \( \ell^p \) which is divergent with respect to the 2-norm \( \|\cdot\|_p \), but convergent with respect to the 2-norm \( \|\cdot\|_{2,v} \).

**Example 4.1** Let \( x^{(n)} := e_n \in \ell^p \), where \( e_n = (0, \ldots, 0, 1, 0, \ldots) \) (the 1 is the \( n \)th term) and let \( \{z_1, z_2\} \) be a linearly independent set where \( z_1 = (1, 0, \ldots) \) and \( z_2 = (0, 1, 0, \ldots) \), then \( \|x^{(m)} - x^{(n)}, z_i\|_p = 2^\frac{n}{p} \to 0 \) as \( m, n \to \infty \). Since \( (x^{(n)}) \) is not a Cauchy sequence with respect to \( \|\cdot\|_p \), it is not convergent with respect to \( \|\cdot\|_p \). However, \( \|x^{(n)} - x^{(n)}, z_i\|_{2,v} = |v_i v_n|^\frac{n-2}{p} \to 0 \) as \( n \to \infty \), since \( v_n \to 0 \) as \( n \to \infty \). Hence, \( (x^{(n)}) \) is convergent with respect to the norm \( \|\cdot\|_{2,v} \).

If we wish, we can also define another weighted norm \( \|\cdot\|_{\beta,v} \) on \( \ell^p \), where \( 1 \leq \beta \leq p < \infty \), by

\[
\|x, z\|_{\beta,v} := \left[ \frac{1}{2^{\beta}} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-\beta} \|x_{k_1} z_{k_1}, x_{k_2} z_{k_2}\|^{\frac{\beta}{p}} \right]^{\frac{1}{\beta}}.
\]

Here \( p \) may be less than 2. Note that if \( \beta = p \), then \( \|\cdot\|_{\beta,v} = \|\cdot\|_p \).

The following proposition gives a relationship between two such weighted 2-norms on \( \ell^p \).

**Proposition 4.3** Let \( 1 \leq \beta < \gamma \leq p \). Then we have

\[
\|x, z\|_{\beta,v} \leq 2^{\frac{1}{p}} \|v\|_p \|x, z\|_p \|x, z\|_{\gamma,v}
\]

for every \( x, z \in \ell^p \).
Proof. Suppose that \( x, z \in \ell^p \). We compute

\[
\|x, z\|_{\beta, v} = \left[ \frac{1}{\beta} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}| \right]^{\frac{1}{\beta}} = \left[ \frac{1}{\beta} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{\frac{p}{\gamma}} \right]^{\frac{1}{\beta}} \leq \left( \frac{1}{\beta} \right)^{\frac{1}{\beta}} \left( \frac{1}{\beta} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{\frac{p}{\gamma}} \right)^{\frac{1}{\beta}} = 2^{\frac{1}{\beta}-\frac{1}{\gamma}} \|v\|_p^{\frac{2(p-\beta)}{\gamma}} \|x, z\|_{\gamma, v},
\]

as desired. \( \blacksquare \)

Corollary 4.4 If \( 1 \leq \beta < 2 < \gamma \leq p \), then there are constants \( C_{1,v}, C_{2,v} > 0 \) such that

\[
\|x, z\|_{\beta, v} \leq \|x, z\|_{2,v} \leq C_{2,v} \|x, z\|_{\gamma, v}
\]

for every \( x, z \in \ell^p \).

5 Further Results

Let \( 2 < p < \infty \). We observe from [2] that \( \ell^p \subset \ell_2^v \), as sets and the inclusion is strict. With respect to the 2-norms on these spaces as we have seen in the previous section, every \( x, z \in \ell^p \) have \( \|x, z\|_{2,v} < \infty \). This suggests that \( \ell^p \) lives inside a larger space, consisting all \( x, z \) with \( \|x, z\|_{2,v} < \infty \).

Proposition 5.1

1. If \( x, z \in \ell^p \) with respect to \( \|x, z\|_p < \infty \), then \( x, z \in \ell_2^v \) with respect to \( \|x, z\|_v < \infty \).

2. The converse is not true.

Proof. Let \( x, z \in \ell^p \) with respect to \( \|x, z\|_p < \infty \). It follows from (4) that \( \|x, z\|_{2,v} \leq 2^{\frac{1}{\beta}-\frac{1}{\gamma}} \|v\|_p^{\frac{2(p-\beta)}{\gamma}} \|x, z\|_p \), which means that \( x, z \in \ell_2^v \) with respect to \( \|x, z\|_{2,v} < \infty \). To show that the converse is not true, we need to find \( \|x, z\|_{2,v} < \infty \) but \( \|x, z\|_p = \infty \). We know that for all \( v_k > 0 \) for all \( k \in \mathbb{N} \), and \( v_k \to 0 \) as \( k \to \infty \). So, choose \( m_1 \) such that \( v_{m_1}^{p-2} < \frac{1}{2}, m_2 > m_1 \) such that \( v_{m_2}^{p-2} < \frac{1}{2} \), \( m_3 > m_2 \) such that \( v_{m_3}^{p-2} < \frac{1}{4} \), and so on. Since the process never stops, we obtain an increasing sequence of nonnegative integers \( (m_j) \) such that \( v_{m_j}^{p-2} < \frac{1}{2^j} \) for every \( j \in \mathbb{N} \). Now, put \( x_k = 1 \) for \( k = m_1, m_2, m_3, ..., x_k = 0 \) otherwise. Let \( \{z_1, z_2\} \) be a linearly independent set where \( z_1 = (1, 0, ...) \) and \( z_2 = (0, 1, 0, ...) \). Hence for \( i = 1, 2 \),

\[
\|x, z_i\|_{2,v}^2 = \frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} \frac{x_{k_1} x_{k_2}}{z_{ik_1} z_{ik_2}} = \sum_k |v_{k_1} v_{k_2}|^{p-2} x_k^2 = \sum_k |v_{ik}^{p-2} \sum_k v_{m_j}^{p-2} < \infty.
\]
while for \( i = 1, 2 \)

\[
\|x, z_i\|_p^p = \frac{1}{2} \sum k_1 \sum k_2 |x_{k_1} z_{ik_1} - x_{k_2} z_{ik_2}|^p = \sum_{k_1} |x_{k_1}|^p = \sum_{k_{i, j}} 1 = \infty.
\]

\( \blacksquare \)

**Lemma 5.2** \([2]\) The space \((\ell_v^2, \| . \|_{2,v})\) is complete with \(\| . \|_{2,v} := \left( \sum_{k} v_k^{p-2} |x_k|^2 \right)^{\frac{1}{2}}\). Accordingly, \((\ell_v^2, \langle \cdot, \cdot \rangle_v)\) is a Hilbert space.

**Proposition 5.3** For every \( x, y, z \in \ell_v^2 \), we have

\[
\|x, y\|_{2,v}^2 + \|x, z\|_{2,v}^2 \leq \left( \|y\|_{2,v}^2 + \|z\|_{2,v}^2 \right) \|x\|_{2,v}^2.
\]

**Proof.** The proof can be done similarly as in \([5]\). \( \blacksquare \)

**Lemma 5.4** For any linearly independent set \( \{a, b\} \) in \(\ell_v^2\), we have

\[
\frac{4 \|a, b\|_{2,v}^2 \|x\|_{2,v}^2}{9 (\|a\|_{2,v}^2 + \|b\|_{2,v}^2)} \leq \|x, a\|_{2,v}^2 + \|x, b\|_{2,v}^2 \leq \left( \|a\|_{2,v}^2 + \|b\|_{2,v}^2 \right) \|x\|_{2,v}^2.
\]

**Proof.** The proof can be done similarly as in \([5]\). \( \blacksquare \)

This shows that

\[
\frac{2 \|a, b\|_{2,v}}{3 \left( \|a\|_{2,v} + \|b\|_{2,v} \right)} \|x\|_{2,v} \leq \|x\|^* \leq \left( \|a\|_{2,v}^2 + \|b\|_{2,v}^2 \right)^{\frac{1}{2}} \|x\|_{2,v}
\]

where

\[
\|x\|^* = \left( \|x, a\|_{2,v}^2 + \|x, b\|_{2,v}^2 \right)^{\frac{1}{2}}.
\]

It is easy to see that \(\|x\|^*\) is a norm. Hence, we can say that \(\|x\|^*\) and \(\| . \|_{2,v}\) are equivalent. This result can also be used to understand the topology of \(\ell_v^2\) as a 2-normed space and it will facilitate the process to show that \(\ell_v^2\) is a 2-Banach space. As similar results in \([5]\) by this way, it can be possible to compute the length in a 2-normed space.

**Corollary 5.5** If \((x^{(n)})\) is a Cauchy sequence in \(\ell_v^2\) with respect to the \(\| . \|_{2,v}\), then it is also a Cauchy sequence with respect to the \(\| . \|_{2,v}\), if it is convergent with respect to the \(\| . \|_{2,v}\), then it is also convergent with respect to the \(\| . \|_{2,v}\).

**Proof.** It is easy to see from the previous lemma. \( \blacksquare \)

**Corollary 5.6** The space \((\ell_v^2, \| . \|_{2,v})\) is complete. In other words, \(\ell_v^2\) is a 2-Banach space with respect to the \(\| . \|_{2,v}\). Accordingly, \((\ell_v^2, \langle ., . \rangle)\) is a 2-Hilbert space.

**Proof.** We know from Lemma 5.2 that \((\ell_v^2, \| . \|_{2,v})\) is complete. Following Corollary 5.5, Lemma 5.4, the desired result is obtained. \( \blacksquare \)

We know that \(\ell^p\) is complete when it is equipped with the usual 2-norm \(\| . \|_p\). The following proposition says that it is no longer so when it is equipped with \(\| . \|_{2,v}\).
Proposition 5.7 As a subspace of $\ell^2_v$, $\left(\ell^p_v, \|\cdot\|_{2,v}\right)$ is not complete, but dense in $\ell^2_v$.

Proof. Since $\left(\ell^2_v, \|\cdot\|_{2,v}\right)$ is complete, it suffices to show that $\ell^p$ is not closed in $\ell^2_v$. As in the proof of Proposition 5.1, we construct an increasing sequence of nonnegative integers $(m_j)$ such that $v_{m_j}^{-p} < 2^{-j}$ for every $j \in \mathbb{N}$. Next, for each $j \in \mathbb{N}$, we define $x^{(n)} = (x_k^{(n)})$ by $x_k^{(n)} = 1$ for $k = m_1, m_2, \ldots, m_n$ and $x_k^{(n)} = 0$ otherwise. Let $\{z_1, z_2\}$ be a linearly independent set where $z_1 = e_1 = (1, 0, \ldots)$ and $z_2 = e_2 = (0, 1, 0, \ldots)$. Then we see that $(x^{(n)})$ forms a Cauchy sequence in $\ell^2_v$ since for $m > n$ we have

$$\|x^{(n)} - x^{(m)}, z_i\|_{2,v}^2 = \frac{1}{2} \sum_{k=1}^n \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} \left| x_{k_1}^{(n)} - x_{k_1}^{(m)} \right|^2 z_{ik_1} \leq |v_i|^{p-2} \sum_{j} v_{m_j}^{p-2} \leq 0,$$

as $m, n \to \infty$ for each $i = 1, 2$. Since $\ell^2_v$ is complete, $(x^{(n)})$ is convergent and we know that the limit is the sequence $x = (x_k)$ where $x_k = 1$ for $k = m_1, m_2, m_3, \ldots$ and $x_k = 0$ otherwise. While $x^{(n)} \in \ell^p$ for every $n \in \mathbb{N}$, the limit $x \notin \ell^p$. This shows that $\ell^p$ is not closed in $\left(\ell^2_v, \|\cdot\|_{2,v}\right)$. The fact that $\ell^p$ is dense in $\left(\ell^2_v, \|\cdot\|_{2,v}\right)$ is easy to see, since every $x = (x_k) \in \ell^2_v$ can be approximated by $x^{(n)} := (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ for sufficiently large values of $n \in \mathbb{N}$ with $\|x^{(n)} - x, e_1\|_{2,v} \to 0$ and $\|x^{(n)} - x, e_2\|_{2,v} \to 0$ as $n \to \infty$. ■

Proposition 5.7 motivates us to study $\ell^2_v$ further as the ambient space, replacing $\ell^p$. So far, we have fixed the weight $v = (v_k)$. We now would like to know how the space $\ell^2_v$ depends on the choice of $v$.

Let $V_p$ be the collection of all sequences $v = (v_k) \in \ell^p$ with $v_k > 0$ for every $k \in \mathbb{N}$. Let $v = (w_k), w = (w_k) \in V_p$. We say that $v$ and $w$ are equivalent and write $v \sim w$ if and only if there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 v_k \leq w_k \leq C_2 v_k$$

for every $k \in \mathbb{N}$.

Theorem 5.8 Let $v, w \in V_p$. Then, the following statements are equivalent:

1. $v \sim w$.
2. There exists constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \|x, z\|_v \leq \|x, z\|_w \leq C_2 \|x, z\|_v \quad x, z \in \ell^p.$$

Proof. The chain of implication (1) $\Rightarrow$ (2) is clear. Hence it remains only to show that (2) $\Rightarrow$ (1). Assume that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \|x, z\|_v \leq \|x, z\|_w \leq C_2 \|x, z\|_v \quad x, z \in \ell^p.$$

Take $x := e_n$, where $n \in \mathbb{N}$ is fixed but arbitrary and take a linearly independent set $\{z_1, z_2\}$ where $z_1 = (1, 0, \ldots)$ and $z_2 = (0, 1, 0, \ldots)$. Then $x, z_1, z_2 \in \ell^p$, so that $x, z_1, z_2$ are in $\ell^2_v$ as well as in $\ell^2_w$. Moreover for each $i = 1, 2$,

$$\|x, z_i\|_{2,v} = |v_n v_i|^{\frac{p-2}{2}} \quad \text{and} \quad \|x, z_i\|_w = |w_n w_i|^{\frac{p-2}{2}}.$$
Hence, from our assumption, we obtain

\[ C_1 |v_n v_i|^{\frac{p-2}{2}} \leq |v_n w_i|^{\frac{p-2}{2}} \leq C_2 |v_n v_i|^{\frac{p-2}{2}}, \]

and this holds for every \( n \in \mathbb{N} \). Taking the \( (\frac{p-2}{2}) \)-th roots, we conclude that \( C_1' v_n \leq w_n \leq C_2' v_n \) where \( C_1' = \frac{C_1^{p-2}}{w_i} > 0 \) and \( C_2' = \frac{C_2^{p-2}}{w_i} > 0 \), for each \( i = 1, 2 \). This completes the proof.

\[ \square \]

6 Concluding Remarks

We have shown that the space \( \ell^p \) can be equipped with a (weighted) 2-inner product and its induced 2-norm. Using the 2-inner product, one may define orthogonality on \( \ell^p \).

There we might also be interested in bounded bilinear 2-functionals. For example, for \( 2 < p < \infty \), the 2-functional

\[
F_z (x_1, x_2) = \begin{vmatrix}
\langle x_1, z_1 \rangle_{2,v} & \langle x_1, z_2 \rangle_{2,v} \\
\langle x_2, z_1 \rangle_{2,v} & \langle x_2, z_2 \rangle_{2,v}
\end{vmatrix}
\]

\[
= \frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} \begin{vmatrix}
x_{1k_1} & x_{1k_2} \\
x_{2k_1} & x_{2k_2}
\end{vmatrix}
\begin{vmatrix}
z_{1k_1} & z_{1k_2} \\
z_{2k_1} & z_{2k_2}
\end{vmatrix}
\]

is bilinear and bounded on \( \ell^2_{2,v} \), and its norm can be given by

\[
\|F_z\| = \sup_{\|x_1, x_2\|_{2,v} \neq 0} \frac{|F_z (x_1, x_2)|}{\|x_1, x_2\|_{2,v}}
\]

or

\[
\|F_z\| = \sup_{\|x_1\|_{2,v} \neq 0, \|x_2\|_{2,v} \neq 0} \frac{|F_z (x_1, x_2)|}{\|x_1\|_{2,v} \|x_2\|_{2,v}}.
\]

Clearly \( \|F_z\| \leq \|z_1, z_2\|_{2,v} \), and by taking \( x_i := \frac{z_i}{\sqrt{\|z_i\|_{2,v}}} \) we obtain \( \|F_z\| = \|z_1, z_2\|_{2,v} \). Moreover, we can prove an analog of the Riesz-Fréchet Theorem (see [1]), which states that for any bounded bilinear 2-functional \( g \) on \( \ell^2_{2,v} \), there exists a a linearly independent set \( \{z_1, z_2\} \in \ell^2_{2,v} \) such that

\[
G (x_1, x_2) = \begin{vmatrix}
\langle x_1, z_1 \rangle_{v} & \langle x_1, z_2 \rangle_{v} \\
\langle x_2, z_1 \rangle_{v} & \langle x_2, z_2 \rangle_{v}
\end{vmatrix}
\]

for every \( x_1, x_2 \in \ell^2_{2,v} \) and \( \|G\| = \|z_1, z_2\|_{2,v} \). However, we cannot show the uniqueness of such a set \( \{z_1, z_2\} \).

We have first discussed \( \ell^p \) and its natural 2-inner product and then we can generalize the results
for all \( n > 2 \). In this regard,

\[
\|x, z_2, \ldots, z_n\|_{2,v}^2 = \sqrt{\langle x, x | z_2, \ldots, z_n \rangle_v} = \left[ \frac{1}{n!} \sum_{k_1} \ldots \sum_{k_n} |v_{k_1} \ldots v_{k_n}|^{p-2} |x_{k_1} z_{2k_1} \ldots z_{nk_1}| \right]^2
\]

is an \( n \)-norm derived from \( n \)-inner product on \( \ell^p \).

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References


